ISBN: 978-81-941567-9-6
January 3-6, 2020

epiSTEME 8

EXPLORING STUDENTS' ALGEBRAIC REASONING ON QUADRATIC EQUATIONS: IMPLICATIONS FOR SCHOOL-BASED ASSESSMENT

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The first encounter with abstract mathematical reasoning impedes the understanding of algebra for most students at lower secondary education levels, which continues to upper levels and beyond. A descriptive survey research involving 300 grade 11 students of ages 14 to 20 clustered into low, moderate, and high academic performance was conducted. Written responses were collected using a Mathematical Reasoning Test, regarding students' argumentation modes for justifying an algebraic conjecture and assessment of suggested solutions for a given quadratic equation. Chi-square test revealed no significant relationship between students' modes of justification and the type of school they came from, $c^2(4) = .50$, $p = .97$. However, the majority exhibited limited forms of understanding and comprehension of quadratic equations. There is a serious need for more attention to students' algebraic reasoning in school-based assessment of mathematical learning.

Keywords: *Algebraic reasoning; School-based assessment; Quadratic equations.*

INTRODUCTION

Students' success in secondary school mathematics is partly dependent on their understanding of algebra. This could be attributed to the fact that algebraic reasoning allows students to explore the structure of mathematics (Ontario Ministry of Education, 2013). In the Zambian curriculum for secondary school mathematics (Curriculum Development Centre, 2013), algebra is introduced to students at the beginning (grade 8) of their secondary education. At that level, students are expected to begin developing abstract thinking that may be required for their advancement in mathematics and science subjects. However, being their first encounter with abstract mathematical reasoning, understanding of algebra has proved to be a 'thorn in the flesh' for most students at that level. Through personal experience, Greer (2008) narrates:

It is a troubling experience to sit beside an eighth grader who is vainly trying to remember what to do with an algebraic equation and reflect that several more years of frustration lie ahead for that student (p.423).

The narration above partly suggests that such difficulties might be carried over till the end of their secondary education and later at college or university. This is also evident by reports and studies that have highlighted students' limited understanding and comprehension of algebraic concepts at both secondary school (Examinations Council of Zambia, 2018) and tertiary (Mukuka & Shumba, 2016) levels of education. Deriving and solving quadratic equations has been reported by the Chief examiner (Examinations Council of Zambia, 2016; 2018) as being challenging to most of the candidates who sat for Grade 12 national examinations. Besides that, it has been noted that students' difficulties in comprehending algebraic concepts is not unique to Zambia because similar results have been reported in other settings (see Kramarski, 2008; Lucariello, Tinec, & Ganleyd, 2014; Organisation for Economic Co-operation and Development [OECD], 2013; Susac, Bubic, Vrbanc, & Planinic, 2014).

Despite this being the case, none of the research conducted in Zambia has attempted to understand students' algebraic reasoning at secondary school level. Our belief is that understanding students' difficulties relating to algebra at grade 11 level would give teachers enough time to correct the situation before those students complete their secondary school education. In an attempt to addressing those challenges, more efforts especially in the developing world like Zambia could be channelled towards identifying classroom practices that can foster students' algebraic reasoning rather than focusing on promoting memorisation of facts. Greer (2008) and Kaput (1999) have highlighted the forms of algebraic reasoning that are relevant to schools and how school algebra can be taught. Nevertheless, our intention here is to understand students' algebraic reasoning on quadratic equations because none of the studies conducted in Zambia has made such an attempt. Consequently, results of this study will lay a foundation for further research on how the reasoning abilities could be enhanced among learners of algebra and mathematics in general.

PURPOSE OF THE STUDY

This study seeks to understand students' ability to reason logically and to rationalise and/or justify mathematical claims. This paper reports on the results of the analysis of data that was collected from grade 11 students on a mathematical reasoning test involving quadratic equations and functions. In line with the model recently developed by Jeannotte & Kieran (2017) on the "process aspects" of mathematical reasoning, our analysis is guided by the following research questions:

i. What are the modes of argumentation used by students in justifying algebraic conjectures?
ii. How do the students assess and validate other people's solutions of a given quadratic equation?

These questions explore how students reason algebraically. Understanding how students assess and validate other people's solutions of a given equation is important to anticipate how they can generate their own solutions when they encounter similar tasks during school-based or external assessment of mathematical learning.

METHODOLOGY

Participants of this descriptive survey research were 300 grade 11 students aged between 14 and 20

(M = 16.24, SD = .98). A Cluster random sampling method was used to select the participants from six public secondary schools within the Ndola district of Zambia. Schools were grouped into three clusters based on their average academic performance (high performing, moderate performing, and low performing). To ensure the representativeness of the sample, two schools were randomly selected from each of the three clusters. At each of the participating schools, one grade 11 class was randomly selected and all the students from each of the selected classes were included in the sample. Before administration of the questionnaire, permission from the relevant authority (Ministry of General Education Permanent Secretary, Provincial Education Officer, and the District Education Board Secretary) was sought and granted.

All the participants provided written consent and the study had received ethical approval from the Research and Innovations unit of the College of Education, University of Rwanda. Students' written responses were collected via a '*Mathematical Reasoning Test*' comprising of seven (7) mathematical tasks on quadratic equations and functions. However, this paper focuses on two of those tasks. Task one is concerned with students' justifications about the truth of the statement "$x^2 + 1$ *can never be zero if* $x \in R$"**.** Task two tested the students' ability to assess and select the most convincing solution of a quadratic equation $(x+2)(x+3)=14$. The development and analysis of these tasks were in line with the requirements of the Zambian curriculum for secondary school mathematics (Curriculum Development Centre, 2013) and previous studies (Brodie, 2010; Jeannotte & Kieran, 2017) on students' mathematical reasoning. All the tasks were assessed and validated by mathematics educators at different levels.

Students' written responses were analysed into categories of meaning using descriptive statistics. These categories focused on empirical or inductive reasoning versus analytical or deductive reasoning. Justification through inductive reasoning was based on citing numerical values to expressions or giving examples of numbers that can satisfy a given statement or expression. On the other hand, analytical justifications were based on logical deductions to arrive at a valid generalisation of a given algebraic statement or argument. A contingency table (cross-tabulation) and Pearson Chi-square Test were also generated to determine whether students from high performing schools answered questions differently from others. Factors that influenced the modes of argumentation by students were also identified to provide guidance on the potential areas of focus in future studies.

RESULTS

The algebraic reasoning being implied here is linked to students' ability in making justified inferences with conjecturing, generalisation and justification being central to the reasoning process (Mata-Pereira & da Ponte, 2017). In that respect, students' algebraic reasoning was assessed based on two categories namely, inductive and deductive reasoning.

Students' argumentation modes for justifying an algebraic conjecture

Students were provided with the following statement:

"$x^2 + 1$ *can never be zero*".

Students were then asked to state whether the statement was true or false for real values of *x* and to construct valid explanations to justify their choices. Two hundred thirty-seven (237) students representing 79% agreed that the statement was true and only 33 (11%) indicated that the statement was false, while 30 (10%) of the students did not respond to the question. All the students who indicated that the statement was false attempted to substitute –1 for *x*. In their own thinking "$-1^2 + 1 = 0$" This reflects students' inadequate understanding about substituting numerical values in a given algebraic expression and their failure to square negative numbers.

A follow-up analysis of 237 submissions representing 79% of students who concurred that the statement was true revealed categories of meaning displayed in Table 1. Results show that the majority (n = 120 or 51%) justified their choice with explanations that were out of context. This was followed by those who gave explanations that were classified as inductive reasoning (n = 83 or 35%) while very few (n = 34 or 14%) argued deductively.

A further qualitative analysis of students' explanations indicated that 90 (75%) of those whose justifications were classified as "out of context" had misconceptions about real numbers. They treated real numbers as mere natural numbers. The remaining 30 (25%) of the respondents gave different explanations without any reference to real numbers. The following submissions by two of the respondents reflect such misconceptions: Respondent 1: The statement is true because real numbers are all positive like 1,2,3,4, etc. Picking any number and add 1 cannot give zero.

Respondent 2: The statement is true because of addition of 1. If it was subtraction, it can be zero because 1-1 = 0.

On the other hand, those who argued inductively justified their choices by citing specific integers or natural numbers to inform their conclusions. It was also established that all of those who argued inductively appeared to have mistaken real numbers for integers or natural numbers because none of them cited other forms of rational numbers (such as decimals or common fractions) or irrational numbers.

Among the 14% who were able to justify deductively, 28 (82%) of them explained that whatever real number may be substituted for *x* whether negative or positive, the square of such a number will always be positive. When that positive number is added to 1, the result will always be greater or equal to 1. The remaining 6 (18%) of those respondents made an assumption that $x^2 + 1 = 0$. When they tried to solve this equation, they reached a stage where they could not compute the square root of –1 and concluded that $x^2 + 1$ can never be zero for real values of *x*.

Table 1 also displays the variations in students' argumentation modes (deductive, inductive and out of context) across the three clusters of school average performance levels. Results indicate that 67 (69%) out of 97 respondents from low performing schools agreed that the statement was true. Of this number, 36 (53.7%) justified their choices with explanations that were classified as out of context while 22 (32.8%) justified inductively and only 9 (13.4%) used the deductive mode of argumentation.

Students' argumentation modes		**School average performance levels**			**Total**
		low	*moderate*	*high*	
Out of context	*Count*	36	47	37	120
	% within school average performance level	53.7%	50.5%	48.1%	50.6%
Inductive	*Count*	22	33	28	83
	% within school average performance level	32.8%	35.5%	36.4%	35.0%
Deductive	*Count*	9	13	12	34
	% within school average performance level	13.4%	14.0%	15.6%	14.3%
Total responses	*Count*	67	93	77	237
Total sample		**97**	**122**	**81**	**300**

Table 1: Students' argumentation modes according to school average performance level

Results further indicate that 93 (76%) of the 122 respondents from moderate performing schools agreed that the statement was true. Forty-seven (or 50.5%) of those respondents gave explanations that were classified as out of context while 33 (35.5%) argued inductively and only 13 (14.0%) used a deductive mode of argumentation. Finally 77 (95.1%) of the 81 respondents from high performing schools agreed that the statement was true. Among those responses, 37 (48.1%) justifications were out of context while 28 (36.4%) used inductive reasoning and only 12 (15.6%) used deductive reasoning.

Overall these results indicate that the majority of students from high performing schools (95.1%) rightly recognised the statement as valid compared to 76% of those from moderate performing schools and 69% of the respondents from low performing schools. Based on the results displayed in Table 1, a Pearson Chi-square test was performed to find out whether there was any significant relationship between the way students justified their choices and the type of school they came from. The Pearson Chi-square test in SPSS version 20 showed no significant association between the two categorical variables, $\chi^2(4) = .50, p = .97$. Although many other factors might have contributed to students' inadequate understanding of quadratic equations, it suffices to say that the way teachers taught and assessed them might have led to such a quality landscape in providing valid mathematical justifications regardless of the school average performance level from which respondents were drawn.

Assessment of the suggested solutions for a given quadratic equation

Respondents were presented with the following solutions for a quadratic equation $(x+2)(x–3)=14$.

Solution A

$$(x+2)(x-3)=14$$
$$x+2=14 or x-3=14$$
$$x=14-2 or x=14+3$$
$$x=12 or x=17$$

Solution B

$$(x+2)(x-3)=14$$
$$x^2-3x+2x-6=14$$
$$x^2-x-6=14$$
$$x^2-x-20=0$$
$$x^2-4x+5x-20=0$$
$$x(x-4)+5(x-4)=0$$
$$(x+5)(x-4)=0$$
$$x+5=0 or x-4=0$$
$$x=-5 or x=4$$

Solution C

$$(x+2)(x-3)=14$$
$$x^2-3x+2x-6=14$$
$$x^2-5x-6=14$$
$$x^2-5x=20$$
$$x^2-5x-20=0$$
$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$
$$x=\frac{-5\pm\sqrt{-5^2-4(1)(-20)}}{2(1)}$$
$$x=\frac{-5+\sqrt{-25+80}}{2}$$
$$x=\frac{-5+\sqrt{55}}{2} or \frac{-5-\sqrt{55}}{2}$$
$$x=1.21 or x=-6.21$$

Table 2

Respondents were then asked to assess each of the three solutions and indicate whether it was correct or wrong. They were also required to justify their choices and to provide their own solutions in an event where they found that all the three given solutions were wrong. Only 17 students representing 6% were able to dismiss all the three solutions and provide valid justifications for their solutions.

Among the 276 students who assessed solution A, 186 (64.4%) rightly identified it as a wrong solution although most of them (97%) could not justify why the solution was wrong. The few (3%) that managed to justify their choices indicated that factors on the left hand side were not supposed to be equated to 14 as that would be the case only if the right hand side was represented by zero. Those who rated solution A as being correct (n = 90 or 32.6%) had a misconception that it was okay to equate each of those factors on the left hand side to 14. They made such a wrong choice even without trying out whether those solutions fitted into the given equation. This could be another indication that students did not understand the property that $x = p$ *or* $x = q$ if and only if $(x–p)(x–q) = 0$ for real numbers p and q.

Of the 263 respondents who managed to assess solution B, 148 (56.3%) of them rightly recognised it as a wrong solution. The common error that was identified in this solution was the "renaming" of the middle term, $–x$ (i.e. $–4x + 5x$ instead of $–5x + 4x$ or $4x – 5x$). It was further established that solution B was the most misinterpreted one because 115 (43.7%) of the respondents recognised it as a correct solution when it was actually not. This clearly shows that those students did not pay attention to arithmetical computations, neither did they try to confirm whether those solutions could satisfy the given equation or not.

Among the 258 respondents who managed to assess solution C, 191 (74%) made the right choice by recognising it as a wrong solution. This shows that a higher proportion of those students managed to recognise what went wrong in the solution. Although a lot of things went wrong in this solution, about 90% of the respondents identified only one error (i.e. $-3x + 2x = -5x$ instead of $-3x + 2x = -x$). Very few (10%) of them went ahead to look at those errors committed when substituting the values of the constants (a, b, and c) in the quadratic formula $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

All of the 67 (26%) respondents that wrongly identified solution C as being correct indicated that the solution was correct because it was the only one that utilised the correct formula for solving quadratic equations. This group of respondents made such a choice without any attempt of looking at how the formula was utilised. It was also quite surprising to learn that this proportion of students at their level (grade 11) could not even see that $-3x + 2x = -5x$ was wrong. This is a confirmation that some difficulties encountered at primary and junior secondary levels regarding integer addition might have persisted even during their senior secondary education.

DISCUSSION AND IMPLICATIONS OF THE FINDINGS

Considering the nature of mathematical tasks presented to participants, they were expected to provide logical justifications and arguments when explaining the validity of the given conjectures or claims. Although this expectation might sound quite odd since it is not stated whether students had been taught to provide such justifications, it is in line with what the curriculum demands (Curriculum Development Centre, 2013). However, responses from a majority of participants reflect an inadequate view of the nature and function of algebraic reasoning in mathematics (Ontario Ministry of Education, 2013; Van de Walle, Karp, & Bay-Williams, 2011). Students' responses to the given tasks were an indication that most of the work given in their classrooms was more of integer solutions to quadratic equations. Classification of the solutions of quadratic equations based on the value of the discriminant seemed to have been rarely discussed in those classrooms. We concur with other scholars (e.g. Brahier, 2016; Brodie, 2010; Small, 2017) that classroom-based assessment should not be limited to memorisation of facts but to enable students to make conjectures and develop formal or informal arguments declaring or supporting why they believe something is true or false.

Additionally, more than half of the students could not justify why the statement "*$x^2 + 1$ can never be zero for the real x*" is true. We found that most of those who failed to construct valid justifications about the truth of this statement had misconceptions about real numbers. The concept of real numbers is usually discussed in grade 8 (Curriculum Development Centre, 2013) but we found that grade 11 students expressed limited understanding of what constitutes real numbers. Substitution of numerical values into a given algebraic equation also proved to be challenging to most students. Some students refuted the algebraic conjecture because of their failure to square negative numbers. They ended up with not knowing that. This quality landscape also reflects teachers' failure to emphasise the importance of signs when manipulating algebraic expressions and number concepts.

On solution validation, refutation and assessment practices, majority (more than 56%) rightly identified the three given solutions to be (as being) wrong but only 6% of them managed to justify why those solutions were wrong and went ahead to provide their own correct solutions. One inference that can be drawn here is that learners might not have been exposed to such kind of questions. We are of the view that asking students to evaluate and validate suggested solutions of different equations is another way through which teachers could understand students' reasoning abilities and their misconceptions of quadratic equations and algebra in general.

It was also established that some of the student errors and misconceptions were not only due to teachers' 'inappropriate' instructional and assessment approaches. We observed that some students' algebraic reasoning abilities were quite low such that it would be difficult for them to solve algebraic tasks requiring higher order thinking. This is why Greer (2008) suggested that "students should be encouraged to study algebra in the spirit of keeping options open, given its status as a gatekeeper to many educational and economic opportunities" (p.427). In other words, failure to pass algebra in school mathematics should not be encouraged, neither should it be an impediment to a student's educational advancement because some careers may require a great deal of algebra while others may only need some elementary algebra.

CONCLUSION

The main sources of student errors have been attributed to the way teachers teach and the way they assess learners. There is a discrepancy between the demands of the curriculum and the way it is implemented. Teaching to make students pass the examinations has accentuated memorisation and recall of facts among students in most Zambian secondary schools. This demonstrates the need to base teaching and assessment methods on evaluations of how students reason algebraically, and on how they communicate that reasoning to others. To reduce the backwash effect of examinations, teachers ought to discuss with students why quadratic equations are important and how knowledge of algebra or mathematics in general would enable them to solve real world problems. A great deal of research is needed to determine how teachers can ensure that learners are conversant with the functions and characteristics of algebra and how such knowledge could be applied in real life situations.

ACKNOWLEDGEMENTS

Special thanks go to the African Centre of Excellence for Innovative Teaching and Learning Mathematics and Science, University of Rwanda, College of Education for their financial and academic support during the conduct of this research.

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