Teacher Efforts Towards the Development of Students' Mathematical Reasoning Skills

Angel Mukuka[1,2] Sudi Balimuttajjo[3] Vedaste Mutarutinya[4]

[1] Department of Mathematics, Science, and Technology Education, Mukuba University, Kitwe, Zambia.
[2]African Centre of Excellence for Innovative Teaching and Learning Mathematics & Science, University of Rwanda - College of Education, Rwanda.
[2] Department of Educational Foundations and Psychology, Mbarara University of Science & Technology, Uganda
[3]Department of Mathematics, Science and Physical Education, University of Rwanda – College of Education (UR-CE), Rwanda.

Corresponding author email: mukukaangel@yahoo.com; https://orcid.org/0000-0003-0292-1324

Abstract

Research shows that enhanced mathematical reasoning leads to increased conceptual understanding and application of mathematical knowledge to various practical situations. However, the identification of classroom practices that support the development of students' mathematical reasoning and the assessment of teacher efforts towards the development of such skills have received little attention in existing research. An explanatory sequential mixed methods research design was employed. Sixty-two teachers of mathematics from six randomly selected public secondary schools within one district completed the questionnaire. Teachers' questionnaire responses were complemented by lesson observations conducted in six randomly selected grade 11 classrooms from all participating schools. Results show that more than 53% of the teachers reported having been employing classroom activities aimed at deepening students' mathematical reasoning skills. Nevertheless, lesson observations revealed some missed opportunities for enhancing students' mathematical reasoning skills. The absence of or little emphasis on thought-provoking tasks, modeling with mathematics, students' exposure to investigating mathematical structures, use of multiple representations, and collaboration among students were observed. These findings demonstrate a need for more professional development opportunities aimed at orienting both in-service and pre-service teachers on effective classroom practices that are bound to deepen students' mathematical reasoning.
Keywords: classroom practices, mathematics classroom, mathematical reasoning skills.

Introduction

Today's world is characterised by high demands and the ever-increasing complexities related to new scientific and technological advancements. As such, individual citizens need to possess relevant skills to solve challenging problems that society is faced with. Mathematics is a discipline that offers tools and processes that may be required for a person's intellectual development. This is why most mathematics curriculum reforms around the world (e.g., Australian Curriculum, 2013; Curriculum Development Centre, 2013; Department of Education, 2003; Ministry of Education, 2007; National Council of Teachers of Mathematics, 2000, 2014; Qualification & Curriculum Authority, 2004) have persistently called for changes designed to provide opportunities for the development of students' mathematical reasoning.

Literature is well-stocked with research confirming that reasoning is a very important component of mathematics and that it is one of the features that distinguishes mathematics from other disciplines (Aricha-Metzer & Zaslavsky, 2017; Mata-Pereira & da Ponte, 2017; National Council of Teachers of Mathematics, 2000; Ross, 1998). Similarly, Hendriana et al (2019) and Anonymous (2020b) have recognised reasoning as one of the critical skills in mathematical problem-solving activities. Likewise, Brito et al (2020) assert that learning mathematics is linked to reasoning abilities. This could be attributed to the fact that reasoning provides the opportunity for students to engage with mathematics at a deeper and thought-provoking intellectual level. Students engaged in mathematical reasoning gain familiarity with the mathematical structure/language that eventually increases their conceptual understanding.

In the context of this study, students' engagement with solving thought-provoking tasks, modelling with mathematics, an examination of mathematical structures, use of multiple representations, and collaboration have been considered, among others as key practices to the development of mathematical reasoning skills. For Battista (1999, p.4), "thinking mathematically inevitably consists of formulating and testing conjectures, making sense of things, and forming and justifying judgments, inferences, and conclusions". He added that a mathematical behaviour is demonstrated by recognising and noticing patterns, or by creating symbol systems with which one can represent, manipulate, and solve problems. This demonstrates that students' enhanced mathematical reasoning leads to increased conceptual understanding and application of mathematical knowledge to real-life situations.

Nonetheless, we wish to express some concern about the little emphasis, in mathematics education literature, on the ways through which teachers of mathematics can teach students to reason logically. There is enough evidence showing that learners of mathematics at different levels of education have exhibited inadequate mathematical reasoning skills (Anonymous, 2020a; Mata-Pereira & da Ponte, 2017). However, there are no studies in Zambia and in many other developing countries that have focused on assessing the status of teacher facilitation skills regarding assessment and enhancement of students' mathematical reasoning. To our knowledge, no research has been conducted to understand how Zambian secondary school teachers are implementing the new curriculum (Ministry of General Education, 2015), which requires students to demonstrate clear mathematical thinking and link the learned content to real-life situations. This is why the present research attempted to investigate the potentially effective classroom practices through which students' mathematical reasoning could be enhanced. It was anticipated that the findings of this study could provide a basis for further interventions aimed at deepening students' mathematical reasoning skills not only in Zambian secondary schools but also in other settings worldwide. With this background, answers to the following research questions were sought:

(i) What efforts do teachers from selected public secondary schools make in developing students' mathematical reasoning skills in the classrooms?

(ii) What are the available opportunities for fostering and assessing students' mathematical reasoning skills in such classrooms?

Fostering Mathematical Reasoning Skills in the Classroom

Mathematical reasoning is a term that encompasses several skills. Most scholars (e.g., Duval, 1991; Ball & Bass, 2003; Jäder, Sidenvall & Sumpter, 2017; Mata-Pereira & da Ponte, 2017) have likened it to a skill that possesses a high deductive - logical quality. Similarly, Lithner (2008) sees mathematical reasoning as a logical sequence of assertions that a learner adopts to reach conclusions in solving a mathematical task. Therefore, taking learners through thought-provoking, challenging, and enriching learning experiences and situations, is bound to develop sound mathematical reasoning. Below are some of the classroom practices and perspectives that could be regarded as being critical to the development of students' mathematical reasoning.

Communication

In a classroom setting, communication is a way through which teachers and students share or clarify ideas. When students are encouraged or challenged to share ideas with others either orally or in writing, they get the hang of being clear, convincing, and decisive in their mathematical expressions. A classroom community that aims at promoting students' mathematical reasoning should not only concentrate on procedural descriptions or mimicking examples but ought to encourage mathematical justification of students' ideas both orally and written. Goos (2004) encourages mathematics classrooms where students are encouraged to participate in discussions in which they can propose and defend arguments, as well as responding to and critiquing or appraising the ideas of their peers or groupmates. Tong et al. (2021) also established that communication of mathematical ideas among students enabled them to be more confident in learning and willing to share their reasoning with others as well as being open to accepting other people's ideas.

Teacher's Talk

In the context of this study, teacher's talk refers to the questions or remarks put forward to the learners during the lesson. The way a teacher frames his/her questions during the lesson plays a significant role in broadening students' mathematical reasoning. Mueller, Yankelewitz, and Maher (2014) postulate that "skillful questioning of student thinking and monitoring of students' problem-solving can provide teachers with a deeper understanding of the development of students' mathematical ideas and help advance students' mathematical growth". This demonstrates that a teacher's talk or questioning should aim at helping learners to reflect on and build upon the existing knowledge as they create new knowledge.

Multiple Representation

In mathematics teaching, multiple representations are used for understanding, describing, developing as well as communicating different mathematical features of the same concept. Multiple representations may include words, symbols, diagrams, formulas, grids and tables, manipulatives, graphs, pictures, and many more. Some national standards (e.g. Qualification & Curriculum Authority, 2004; National Council of Teachers of Mathematics, 2000) have stressed that gaining access to mathematical representations enables students to acquire a set of thinking tools that expands their ability to interpret different aspects of physical and social environments. Duval (2006) also found that the use of different representations for a particular concept leads to learners' increased conceptual understanding. In acknowledging the benefits of multiple representations on students' mathematical reasoning, Dreher, Kuntze, and Lerman (2015) advised that students ought to be supported and encouraged to construct meaning concerning different mathematical representations and make connections among various representations.

Modelling with Mathematics

Literature is well-stocked with studies that have persistently advocated for making mathematics education to be realistic (see Duchhardt, Jordan, & Ehmke, 2015; Gravemeijer, Stephan, Julie, Lin, & Ohtani, 2017). It has been recommended that school mathematics should be relevant to the workplace and real-world experiences. Barker, et al. (2004), also regard modelling with mathematics as being concerned with lessons that help students to apply the mathematics they know in solving real-world problems. Through modelling, students can see mathematical connections among different topics or subjects and the application of mathematical knowledge to real-life situations (National Council of Teachers of Mathematics, 2000). For instance, a topic like 'functions' should aim at increasing students' reasoning ability in relating one quantity of interest to another. Similarly, we may be aware that the larger part of the information that we come across in our daily life situations may need statistical reasoning for their analysis and interpretation. As such, allowing students to put real-world experiences into mathematical terms or relating mathematical content to various practical situations is bound to enhance their mathematical reasoning skills.

Task Selection

This is an activity that is planned before the teacher presents a lesson to the learners. It is a very essential aspect of stimulating students' mathematical reasoning. Mueller et al. (2014) pointed out that "open-ended and thought-provoking tasks initiate specific moves to promote reasoning and understanding among students. Such tasks encourage students to begin to build their justifications and sharing of ideas" (p.3). Other scholars (e.g., Kwon & Capraro, 2021; Jäder et al., 2017; Mata-Pereira & da Ponte, 2017) have also stressed the need for teachers to pose tasks that would trigger higher-order thinking among learners because their reasoning depends on the tasks they encounter in the classroom.

Methodology

Research Design and Participants

Data to address the aforementioned research questions were collected from six public secondary schools in Ndola district of Zambia. An explanatory sequential mixed methods research design (Cresswell, 2014) was employed in which teachers' written responses were collected using a closed-ended questionnaire. Sixty-two teachers of mathematics completed the questionnaire. After questionnaire administration, mathematics lesson observations were carried out in one randomly selected grade 11 class from each participating school. The mixed methods research approach was considered favourable as it provided an opportunity to collect both quantitative and qualitative data to gain deeper insights into the prevailing situation. Cresswell (2014) highlights that a mixed methods research approach has the ability to provide an in-depth and complete understanding of the prevailing situation as opposed to undertaking either a qualitative or quantitative research approach.

Development and Validation of Instruments

Both Teachers' Questionnaire and a Mathematics Classroom Observation Protocol ($MCOP^2$) were used to collect data.

Questionnaire for teachers

After the formulation of questionnaire items, a pilot study was conducted on 33 teachers of mathematics from 4 secondary schools that did not participate in the actual study. Of the 33 teachers, 17 (53.1%) were male while 15(46.9%) were female and 1 teacher did not indicate the gender. Years of mathematics teaching experience at secondary school level ranged from 1 to 28 (M = 8.79, SD = 5.79). This sample (n = 33) was considered sufficient for both an exploratory factor analysis (EFA) and reliability analysis. In his analysis of sample sizes of 30, Yurdugül (2008) found that Cronbach's alpha coefficients were decisive with condition that the first eigenvalue of the Principal Component Analysis (PCA) was more than 6.

It also suffices to state that this paper focuses only on two of the four sections of the questionnaire. That is, demographic information (Part I) and the teachers' existing classroom practices on the development of students' mathematical reasoning (Part II). To establish the construct validity of the items concerning the development of students' mathematical reasoning, exploratory factor analysis was performed using the principal component extraction method in SPSS version 21.

To optimize the number of factors, the default number in SPSS given by Kaiser's criterion (eigenvalue > 1) was used. Four (04) components were extracted and none of the 10 items had a communality score of less than 0.2. Despite that, a rotated component matrix showed that only one component had more than 3 items with factor loadings greater than 0.4. Following recommendations by different scholars (e.g. Child, 2006; Field, 2013; Guadagnoli & Velicer, 1988), the principal component analysis was repeated with one fixed factor and all the 10 items were loaded. At this point, only 7 items had factor loadings greater than 0.4 and were all retained. Table 1 displays the data analysis code and a full statement (as it appears in the questionnaire) for each of the seven retained items. Respondents were requested to indicate the frequency with which they embarked on classroom practices aimed at developing students' mathematical reasoning on a five-point scale rating from 0 (never) to 4 (always)

Table 1. *Classroom Activities aimed at Developing Students' Mathematical Reasoning*

Code	Full Statement
MR1	I encourage the use of mathematics vocabulary terms during mathematics lessons
MR2	I require students to share their thinking with peers by observing patterns, arguing, and justifying ideas
MR3	I encourage students to explain the reasoning behind their ideas both orally and in writing
MR4	I ask students to construct simple algebraic and geometric proofs based on what they already know
MR5	I give tasks that require students to formulate, explore and investigate mathematical conjectures
MR6	I ask students to interpret results from graphs, charts, or tables and the use of multiple representations
MR7	I ask students to relate classroom mathematics to the real-life situations

A further check for multicollinearity indicated that all the retained items were independent since none of the inter-item correlations was above .80 (Field, 2013). Internal consistency of the retained items was also considered sufficient with Cronbach's alpha coefficient of 0.78, which was above the recommended threshold of 0.70 (Adams & Wieman, 2010; Taber, 2018).

Mathematics classroom observation protocol (MCOP^2)

Regarding lesson observations, 12 items for measuring two distinct factors of Teacher Facilitation and Student Engagement were extracted from the Mathematics Classroom Observation Protocol (MCOP^2) that was developed by Gleason, Livers, and Zelkowski (2015). Apart from the order in which the items appear in the original manual and some slight changes in wording, no other modifications were made especially that original authors had already validated the instrument. According to Gleason, et al (2015), "Teacher

Facilitation subscale (α = .850)" was intended to measure teacher-related activities in the classroom while "Student Engagement subscale (α = .887)" measured students' activities in the classroom during the lesson. Although the instrument had been validated by original authors, the adapted observation protocol items were subjected to a pilot study with three observers observing the same lesson. This was done to find out the level of agreement among the observers. The three observers included the researcher and two mathematics teacher educators. The two mathematics teacher educators were selected because of their knowledge of the Zambian mathematics curriculum for secondary schools as well as their experience in monitoring and observing student teachers on teaching practice. Spearman's rank correlation coefficient was used at a 5% level of significance to establish consistency levels among the three observers. It was found that the level of agreement between observers 1 and 2 was significant (r = .83, p < .05). Agreement level between observers 1 and 3 was significant (r = .68, p =.01) and that of observers 2 and 3 was also significant (r = .69, p = .009). These statistics assured that different raters or observers were bound to observe similar classroom activities using the same instrument. Table 2 gives the adapted MCOP^2 items associated with the teacher facilitation and student engagement activities.

Ethical Considerations

Before data collection, research instruments received ethical approval from the Directorate of Research and Innovations of the University of Rwanda, College of Education. Thereafter, permission to collect data from teachers was granted by relevant authorities from the Ministry of General Education in Zambia. All the participants provided written consent before any information was collected from them.

Data Analysis

As pointed out earlier, lesson observations focused on lesson preparation, lesson delivery, and assessment of students' conceptual understanding, mathematical reasoning, and problem-solving. Table 2 and Table 2 illustrate the scores ranging from 0 (not engaged at all to 3 (regularly engaged) indicating the extent to which a particular practice or activity occurred in each of the six 80-minute lessons that were observed. The scoring was done in line with recommendations by the originators (Gleason et al., 2015) of the observation protocol that was adapted. Originators of the instrument also proposed a minimum of three lesson observations for the "would be" users of the protocol. Only quantitative data have been presented in the "results" section as much of the qualitative comments on what transpired in those classrooms have been incorporated in the discussion section. Since similar patterns were observed in most of the lessons that were observed, one lesson on quadratic functions has been discussed as a case scenario.

Results

Teacher Efforts in Promoting Students' Mathematical Reasoning

Respondents were requested to indicate the frequency with which they employed classroom activities aimed at deepening students' mathematical reasoning on a five-point scale rating from 0 (never) to 4 (always). Table 2 illustrates the number of responses and the percentage for each category. Results displayed in Table 2 reflect that very few or none of the respondents indicated that they had never employed any of the classroom practices that are perceived to deepen students' mathematical reasoning skills. It has been further revealed that the majority of respondents had been employing such practices quite often or always. After collapsing the five categories displayed in Table 2 into three categories as displayed in Figure 1, it was noted that 53.2% of the respondents indicated that they had been asking students to relate classroom mathematics to real-life situations (MR7) quite often. Similarly, 57.9% of the respondents indicated having administered tasks that required students to formulate, explore and investigate mathematical conjectures (MR5).

Table 2. *Reported Frequencies Regarding Teachers' Efforts in Developing Students' Mathematical Reasoning*

Teacher Actions	Frequency (N= 62)				
	Always	*Often*	*Sometimes*	*Very rare*	*Never*
MR1	31(50%)	25(40.3%)	4(6.5%)	1(1.6%)	1(1.6%)
MR2	26(41.9%)	26(41.9%)	10(16.1%)	0(0%)	0(0%)
MR3	31(50%)	19(30.6)	10(16.1%)	1(1.6%)	1(1.6%)
MR4	9(14.5%)	34(54.8%)	13(21%)	5(8.1%)	1(1.6%)
MR5	13(21.0%)	24(38.7%)	10(16.1%)	9(14.5%)	6(9.7%)
MR6	23(37.1%)	21(33.9%)	15(24.2%)	1(1.6%)	2(3.2%)
MR7	3(4.8%)	30(48.4%)	21(33.9%)	7(11.3%)	1(1.6%)

Results further revealed that 69.3% had been asking students to construct simple algebraic and geometric proofs based on what they already know (MR4) while 71% had been engaging students to interpret results from graphs, charts, or tables and the use of multiple representations (MR6).

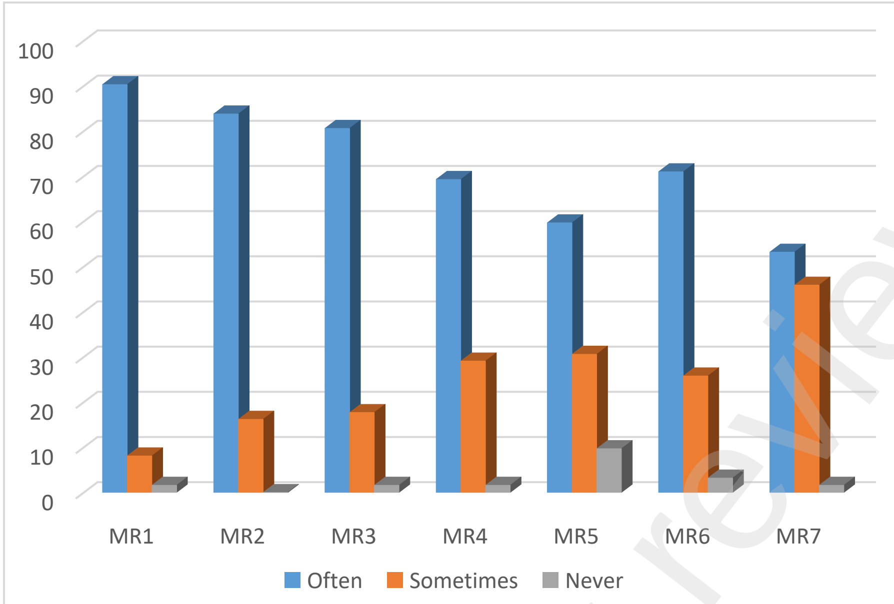

Figure 1. *Reported Frequencies Regarding Teachers' Efforts in Developing Students' Mathematical Reasoning*

Results displayed in Figure 1 further reflect that 80.6% of the teachers had been encouraging their students to explain the reasoning behind their ideas both orally and in writing (MR3) while 83.8 required students to share their thinking with peers by observing patterns, arguing, and justifying ideas (MR2). The highest proportion (90.3%) of teacher respondents was associated with encouraging the use of mathematics vocabulary during mathematics lessons (MR1). The overall impression of the results displayed in Table 2 and Figure 1 is that majority (more than 53%) of the teachers who participated in the study had been making efforts aimed at developing students' mathematical reasoning skills.

Student Engagement and Teacher Facilitation Activities in the Classroom

Table 3 illustrates the scores ranging from 0 to 3 indicating the extent to which a particular practice or activity occurred in each of the six 80-minute lessons that were observed. These numbers give the frequency with which a particular activity occurred during each lesson:

0 = Not engaged at all, 1 = seldom engaged, 2 = sometimes engaged, and 3 = regularly engaged.

Regarding student engagement activities, results displayed in Table 3 indicate that most of the activities seldom occurred during the observed lessons. For instance, none of the six observed lessons had students manipulate multiple representations of mathematical concepts. In situations where at least one representation was used (like in lesson 1), it was exclusively manipulated by the teacher only.

Table 3. Student Engagement and Teacher Facilitation Activities

MCOP^2 Item	Lesson 1	Lesson 2	Lesson 3	Lesson 4	Lesson 5	Lesson 6
Student Engagement Activities						
Exploration, investigation, or problem solving	1	1	1	3	1	1
Multiple representation	1	0	0	0	0	0
Perseverance in problem solving activities	1	2	1	2	1	1
Climate of respect for other people's opinions	2	2	2	2	1	1
Communicating ideas to others (peer-to-peer)	1	3	1	1	0	1
Teacher Facilitation Activities						
Promotes conceptual understanding	2	1	1	3	1	1
Promotes modelling with mathematics	0	0	0	0	0	0
Engages students in examining mathematical structures	0	0	0	3	0	0
Tasks that have multiple paths to a solution	2	2	3	2	1	1
Promotes precision of mathematical language	2	3	2	3	1	1
Promotes higher-order thinking	2	3	1	3	2	2
Climate of respect for other people's opinions	2	2	2	2	1	1

Another significant result from Table 3 is that only lesson 4 had students' regular engagement in exploration, investigation, and/or problem-solving. The rest of the lessons had few students actively engaged while the majority remained passive. Although lesson 2 was characterised by students' regular engagement in communicating their ideas with others, it can be noted that this activity rarely occurred in other lessons and did not even take place in lesson 5. To some extent, this contradicts what teachers reported in the questionnaire (refer to item 2 in Table 2) that they had been encouraging their students to share their thoughts and explain their reasoning with others. Results displayed in Table 3 further indicate that the few students who regularly engaged in exploration, investigation, or problem-solving also showed some perseverance in problem-solving as well as respecting what others had to say.

Concerning teacher facilitation activities, results displayed in Table 3 reflect that most of the observed teachers prepared lessons that promoted precision of mathematical language, and had used students' questions or responses to enhance conceptual understanding. Results further reflect that teachers showed some respect for what learners had to say. Another positive result that was observed in almost all the lessons was that teachers tried to design lessons that incorporated fundamental concepts of the subject to promote conceptual understanding as well as incorporating tasks that had multiple solution paths. While these results are partially consistent with the teachers' questionnaire responses (refer to Table 2), the extent to which these activities occurred in the observed classrooms was not as much as it was portrayed in the questionnaire responses. For instance, none of the observed teachers attempted to engage students in modelling activities. Neither did any of the observed teachers (except for lesson 4) engage students with the examination of mathematical structures (I.e., conjecturing, justifying, and generalising). Further qualitative insights into what transpired in the observed lessons are discussed in the following section.

Discussion

The overall impression from the results displayed in Table 2 and Figure 1 is that the majority of teachers who participated in the study had been making efforts in promoting students' mathematical reasoning in their classrooms. Some responses given by teachers were consistent with what was noticed during the lesson observations. This gives an indication that there is a higher possibility of improvement in the areas that teachers were found lacking. Amidst all the positive and promising results from teachers' questionnaire responses as well as lesson observations, it was further noted that teachers missed several opportunities through which they could have enhanced the reasoning abilities of learners. Notable among those opportunities include the absence or little emphasis on thought-provoking tasks, modelling with mathematics, an examination of mathematical structure (I.e. noticing patterns, conjecturing, justification and generalisation), use of multiple representations, and collaboration among students.

In all the lessons observed, it was noted that much of the tasks given to the learner seemed to be of "lower-order" knowledge-based questions whose responses focused more on memorisation or recall of facts without much emphasis on "how or why" something is true. For instance, a lesson on *circle theorems* only encouraged students to recall theorems and definitions without much emphasis on proving those theorems. This is contrary to the belief that providing students with opportunities to engage in proving theorems does not only help them extend their thinking but also solidifies their understanding of various mathematical concepts and improves their mathematical problem-solving skills (Anonymous 2020a; Thompson et al., 2012).

In another lesson on *"Sketching graphs of quadratic functions",* it was noted that the lesson captured most fundamental concepts (such as x and y-intercepts and the turning point) that are required as prescribed in the syllabus for secondary school mathematics (Curriculum Development Centre, 2013). However, several opportunities for building conceptual understanding and reasoning among students were missed since there was not much emphasis or special focus on the *"why"* behind any of the procedures and facts included. Below are four illustrative examples to explain this:

First, the reasoning behind the placement of zero in place of y when finding the x-intercepts was not clear. Students were only told to put zero in place of y without any reason as to why that was the case. This seemed to promote memorisation of facts without a deeper understanding of the concept.

Second, when finding x-intercepts of the function $y = 10 - 3x - x^2$, a quadratic equation $0 = 10 - 3x - x^2$ was generated and the picture portrayed to students was that such an equation can only be solved using the formula $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$, without any reference to other methods such as factorisation or completing the square. Here, we can see that learners were not encouraged to seek multiple paths to the solution of a problem. Experience shows that a substantial number of learners of school mathematics are unable to recall the exact formula for solving quadratic equations. While the Examinations Council of Zambia has introduced the practice of listing the quadratic formula at the beginning of an exam paper, there is a need for teachers to emphasise the use of other methods (such as completing the square) to avoid over-reliance on one strategy and the backwash effects of examinations (Anonymous 2020b).

Third, students were advised to memorise the formula for finding the turning point as $\left(\frac{-b}{2a}, \frac{4ac - b^2}{4a}\right)$ without deriving it with them. This could have been a good opportunity for the teacher to guide learners on the different ways of obtaining the turning point of the function $y = ax^2 + bx + c$. For instance, the concept of gradient function from calculus, completing the square, or the use of mid-point between the x-intercepts could have been encouraged to allow learners to use a method of their choice. The derivation of the formula could have been done so that students see and comprehend the underlying mathematical ideas instead of memorising facts without a deeper understanding. In light of these findings, we wish to concur with Sears (2018) who recommends that "more opportunities for pre-service teachers to prove be provided in mathematics education pedagogical and content courses" (p.16). This is also consistent with calls by Luneta (2022) on the need to design and provide continuing professional development programmes for in-service teachers to ensure that they remain abreast with current trends in the field to produce the best learning results for their students.

Fourth, there was a need to give tasks that would have required learners to go beyond just drawing the graph but connecting classroom mathematics with real-world experiences. Failure to develop students' reasoning skills in the interpretation of various algebraic concepts makes it difficult for them to interpret similar situations in other subjects. For instance, Turşucu et al (2018) observed that some physics students may not be able to recognise that the displacement formula, $s = \frac{1}{2} at^2$ has a similar algebraic structure as that of any quadratic function of the form $y = ax^2$. This is why it is important to provide opportunities for students to notice patterns for themselves, make conjectures, and draw inferences. Brodie (2010, p.45) provides an example of a task that promotes learners' mathematical reasoning on quadratic functions. Issues of minimum/maximum turning points, shrinking, stretching, and reflections of graphs would be noticed by students themselves instead of asking them to memorise formulas. This might have worked as an advance organiser (Githua & Nyabwa, 2008) for the following lesson that may incorporate concepts of intercepts and turning points because students could have formulated and tested conjectures on their own. Likewise, students' ability to formulate and test conjectures makes them autonomous and moves them away from depending on the teacher as the only authority in the classroom because they can see and observe patterns for themselves (Mueller et al., 2014).

Consistent with what has been reported previously (Anonymous, 2019), it was noted that most lessons were primarily teacher-directed as little or no opportunities were available for student-to-student conversations. Only those students who were called upon to present their solutions on the chalkboard had an opportunity to communicate their ideas with the rest of the class. Otherwise, the majority of students were actively involved in listening to the teacher or fellow student presenting on the chalkboard. This is contrary to the views held by Goos (2004) who encourages classroom communities where students are provided with opportunities to "learn to talk and work together by participating in mathematical discussions, proposing and defending arguments and responding to the ideas and conjectures of their peers" (p.259). A review by Alegre et al (2019) also established that peer tutoring improved conceptual understanding and academic achievement among learners of school mathematics. Similarly, Chazan and Ball (1999) acknowledged that deepening the reasoning ability of students in mathematics classrooms requires that students are provided with opportunities to voice their mathematical thinking to peers during group discussions. Through such classroom discourse, students could reflect on their own and other peoples' views and then correct misconceptions collectively.

The other aspect that was missing in almost all the observed class sessions is modelling with mathematics. One of the popular questions among most learners of mathematics at all levels of education is *"where am I going to use these mathematics after all?"* This is one of the questions that we (teachers) have failed to address at one point or the other in our classrooms. Brodie (2010) pointed out that "learners' inability to see mathematics as a worthwhile human activity is in part due to the low level of mathematical reasoning" (p.57). This low level of mathematical reasoning among our students could be attributed to our failure (as teachers) to design tasks that help students to use classroom mathematics in solving real-world problems (i.e., problems arising from everyday life experiences, society, and workplaces). For example, those lessons should have incorporated tasks that could have enabled students to use knowledge about a quadratic function to determine how one quantity varies with another quantity (Gleason et al., 2015). Likewise, Kwangmuang et al. (2021) recommend a need for the design of lessons that incorporate practical situations to stimulate higher-order thinking among learners.

Finally, this study suggests that how we (teachers) guide our students to reason mathematically is in part reliant upon different ways available for representing a concept. Learners need to make use of different representations of a particular concept instead of watching a teacher manipulating such representations.

Kwon and Capraro (2019) also acknowledge that meaningful problem-posing and the use of multiple representations can help in increasing students' understanding of mathematical concepts. Similarly, Gleason, Livers, and Zelkowsk, (2017) guided that "the representations can be student-generated (a drawing or a graph) or provided by the teacher (manipulatives or a table), but it is the students that must then use the representation" (p.125). However, the results of the present study show that most of the lessons did not involve multiple representations. In instances where multiple representations were included, they were restrictive to teacher manipulation and use. This could have led to students being stuck with only one way of generating a solution to a particular problem even when other paths to arriving at the same solution were available. This is why some scholars (Niyukuri, et al., 2020; Nel and Luneta; 2017) emphasise the importance of designing professional development programmes aimed at supporting teachers of mathematics in what they teach and how to teach it.

Limitations

It is a well-known fact that every research instrument has its limitations. The two research instruments used in the present study are not an exception. Regarding questionnaire responses, some of the respondents (teachers) may have reported something that they do not even practice in real classrooms. This is why lesson observations were carried out. It also suffices to mention that lesson observations in six classrooms might not be enough to warrant a complete understanding of how teachers promoted students' mathematical reasoning in those classrooms. This is why the original authors (Gleason et al., 2017, p.119) of the observation protocol indicated that most of the lessons designed for improving procedural fluency may not have high scores on some of the protocol's indicators. Given these limitations, it is recommended that future studies on this subject should increase the number of questionnaire respondents as well as the number of lesson observations to increase the accuracy of results because larger samples are likely to reduce standard errors. There is also a need for future studies to quantify the extent to which the cited instances or opportunities can promote the mathematical reasoning skills among learners of school mathematics.

Conclusion

This paper has highlighted some of the potentially effective ways through which the students' mathematical reasoning skills could be enhanced. It has been strongly recommended that teachers of school mathematics need to focus on developing students' mathematical reasoning to increase conceptual understanding and application of mathematics to real-world experiences. Among the ways through which students' mathematical reasoning could be enhanced includes; designing and administering thought-provoking tasks, encouraging collaboration among learners, using multiple representations, linking classroom mathematics to real-world experiences, and encouraging a culture of mathematical justification and proof. Teachers could also improve their instruction by collaborating with colleagues and looking at existing lessons to determine how each of the cited opportunities could be incorporated to foster students' mathematical reasoning. Continuing professional development activities have emerged as effective ways through which serving teachers could be supported in topic-specific pedagogical content knowledge. Although this study was contextually bound, its findings are beneficial to all mathematics educators, researchers, and learners worldwide.

Acknowledgement

We wish to express our sincere thanks to all the teachers who participated in the study, and the respective school authorities that permitted our engagement with them. Our sincere gratitude also goes to the Permanent Secretary, Provincial Education Officer, and District Education Board Secretary who authorized the gathering of data from the teachers.

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