Numerical Solution to Two-point Boundary Value Problems with Neumann Boundary Conditions using Galerkin Finite Element Method

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2 authors:


Dominic Tuhimbise
Mbarara University of Science & Technology (MUST)


Ronald Mwesigwa
Mbarara University of Science & Technology (MUST)

Numerical Solution to Two-point Boundary Value Problems with Neumann Boundary Conditions using Galerkin Finite Element Method

Dominic Tuhimbise, Ronald Mwesigwa
Department of Mathematics, Mbarara University of Science and Technology,
P. O. Box 1410, Mbarara, Uganda

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Abstract

In this study, Galerkin finite element method (FEM) has been developed to approximate the solution of both second-order linear with constant and non-constant coefficients, and nonlinear second-order two-point BVP of ordinary differential equations with Neumann boundary conditions. Lagrange linear piece-wise polynomials have been used as basis functions. Linear second order two-point boundary value problem (BVP) of ordinary differential equations (ODEs) with non-constant coefficient was solved by applying Gauss quadrature 3-point rule in the Galerkin FEM. For the nonlinear BVP, the Newton's method was used with the Galerkin FEM. The errors in approximations have been studied, noting that for this method, errors in the approximations reduce with decreasing element or step size. The convergence and consistency of Galerkin FEM applied to the linear and nonlinear second-order boundary value problems of ordinary differential equations have been discussed. The results have been presented in a number of tables and illustrated using graphs, all generated using MATLAB. Basing on the results from the simulations, it was found that the method was stable, convergent and consistent since further reduction of element or step sizes produced significant reduction in the error of all test problems. Thus, the developed method performs well with linear and nonlinear two point BVPs.

Key words: Boundary value problems, finite element method, ordinary differential equations, Neumann boundary conditions.

1 Introduction

Two-point boundary value problems occur in a wide variety of problems, including the modeling of chemical reactions, heat transfer, diffusion, and the solution of optimal control problems. There are several types of BVPs depending on the type of boundary conditions, that is, Dirichlet, Neumann, or mixed. To be useful in applications, BVPs should be well-posed, meaning that, given the input to the problem there exists a unique solution, which depends continuously on the input. Much theoretical work in the field of ordinary and partial differential equations is devoted to proving that BVPs arising from scientific and engineering applications are in fact ill-posed [10]. For example, according to [?], the existence and uniqueness of the solutions of the BVPs with Neumann boundary conditions is guaranteed by the Fréchet-Riesz Theorem and the Lax-Milgram Theorem.

Many important problems in engineering and applied sciences are modeled as second-order two-point BVPs such as the boundary layer theory in fluid mechanics, heat power transmission theory, space technology and reaction kinetics [14]. Since these second-order two-point boundary-value problems have many applications in science and engineering and analytical solutions are usually not available or very complicated to find, then, faster and accurate numerical solutions of such problems are very important [2]. The main objective of numerical analysis is to determine numerical methods to approximate a solution of a real-world problem. Among the numerical methods commonly used to solve second-order two-point boundary value problems of ordinary differential equations, are the shooting method, finite difference method and the finite element method [15, 4]. The solutions to scientific and engineering problems can be obtained more easily using numerical methods with the help of computers.

Some researchers have studied and attempted to find accurate numerical solutions to second order two-point BVPs with different boundary conditions using different numerical methods [16, 14, 8, 18, 6, 3].

Quite a number of researchers have generally come up with many numerical techniques in the field of finite element methods for solving two-point boundary value problems [9, 9]. FEM has become a popular technique for obtaining approximate solutions to the ordinary differential equations and the partial differential equations that arise in science and engineering applications [11, 5, 13]. In [7], Dogan formulated a Galerkin FEM approach for the numerical solution of Burger's equation. A linear recurrence relationship for the numerical solution of the resulting system of ordinary differential equations via a Crank-Nicolson approach involving a product approximation was found. It was shown that this method is capable of solving Burger's equation accurately for a wide range of viscocity values. Sharma et al. [16] proposed the Galerkin FEM for the numerial solution of the two-point BVPs having mixed boundary conditions. The stability of the proposed method is discussed and found that the method is conditionally stable. Zavalani [19] proposed a Galerkin FEM to find the approximate solution of inhomogeneous second order two point BVPs of ordinary differential equations. It was shown that the method is simple, accurate and well-behaved in the presence of singularities. Generally, little attention has been given to second order two-point boundary value problems with Neumann boundary conditions [1].

Much work has been done on solving the second-order two-point boundary problem of ordinary differential equations especially those with either Dirichlet or mixed boundary conditions with different numerical methods, including the shooting method, FDM and FEM. And from the literature that is available, FEM has greatly been applied to finding the numerical solution of BVPs of ordinary differential equations which originate from applications of partial differential equations. This is largely because of its rich origin in engineering. Finite element method has been found to be accurate and efficient when compared to most of the other numerical methods. There seems not to be much published work on solving second order BVPs of ordinary differential equations with Neumann boundary conditions using the finite element methods. Therefore, in this study, the main focus was to solve the second-order two-point BVPs of ordinary differential equations with Neumann boundary conditions using a faster and efficient numerical method of Galerkin-FEM.

This paper is organised in the following manner. In section 2, we formulated the problem using Galerkin finite element method and implemented the proposed method. In section 3, we considered some test problems of linear and nonlinear BVPs with different parameters to illustrate the method and its convergence. In section 4, the results were discussed. The conclusion is then presented in section 5.

2 Problem formulation

In this section, basic steps on how to formulate the boundary value problem using Galerkin FEM are given basing on the problems to be solved, otherwise the general procedure is given in [9],[12].

2.1 The linear second-order BVP

We consider the case of linear second-order two-point boundary value problems with the equation of the form

$$y''(x) + p(x)y'(x) + q(x)y(x) = f(x), \quad \text{for } a \leq x \leq b, \tag{1}$$

with boundary conditions $y'(a) = \alpha$ and $y'(b) = \beta$, and where $p(x) \geq 0$, $q(x) \geq 0$ and $p(x)$, $q(x)$, $f(x)$ are smooth functions on $[a, b]$, α and β are constants. It is assumed that a unique smooth solution, $y(x)$, exists.

Now, to implement the Galerkin FEM to find the solution of the differential equation in (1), the domain $[a, b]$ is subdivided into $N - 1$ sub-domains, called elements that join the nodes $x_1, x_2, \ldots, x_N$, where $a = x_1 < x_2, < \cdots < x_N = b$, and $h = (b - a)/(N - 1)$.
We then let the approximate solution to the exact solution $y(x)$ be

$$u(x) = \sum_{j=1}^{N} u_j \phi_j(x). \tag{2}$$

Where $\phi_j(x)$ is a basis (trial) function. The unknowns u_j are determined by solving a system of N algebraic equations from

$$\int_a^b \phi_i(x)R(x)dx = 0, \tag{3}$$

for $i = 1, 2, \ldots, N$, where $\phi_i(x)$ is the test function and $R(x) = u''(x)+p(x)u'(x)+q(x)u(x)-f(x)$. On substituting for $R(x)$ in (3) and rearranging, we get

$$\int_a^b \phi_i u''(x)dx + \int_a^b \phi_i p(x)u'(x)dx + \int_a^b \phi_i q(x)u(x)dx = \int_a^b \phi_i f(x)dx. \tag{4}$$

Applying integration by parts to the first term, equation (4) becomes

$$\left[\phi_i \frac{du(x)}{dx}\right]_a^b - \int_a^b \phi_i' \frac{du(x)}{dx}dx + \int_a^b \phi_i p(x)u'(x)dx + \int_a^b \phi_i q(x)u(x)dx = \int_a^b \phi_i f(x)dx, \tag{5}$$

In terms of the basis functions of linear lagrange piecewise polynomial,

$$[\phi_i u'(x)]_a^b - \int_a^b \phi_i' \sum_{j=1}^N u_j \phi_j' dx + \int_a^b \phi_i p(x) \sum_{j=1}^N u_j \phi_j' dx + \int_a^b \phi_i q(x) \sum_{j=1}^N u_j \phi_j dx = \int_a^b \phi_i f(x) dx, \quad (6)$$

for $i = 1, 2, \ldots, N$. This is called the weak formulation of the original equation.

Equation (6) is then used to construct an algebraic system of N equations by setting $\phi_i = \phi_1$, up to $\phi_i = \phi_N$, in turn. Applying the boundary conditions in the resulting system, the equation in the first interval of discretisation, may be written as

$$-\sum_{j=1}^N u_j \int_{x_1}^{x_2} \phi_1' \phi_j'(x) dx + \sum_{j=1}^N u_j \int_{x_1}^{x_2} p(x) \phi_1 \phi_j'(x) dx + \sum_{j=1}^N u_j \int_{x_1}^{x_2} q(x) \phi_1 \phi_j(x) dx$$
$$= \int_{x_1}^{x_2} \phi_1 f(x) dx - \phi_1(x_2) u'(x_2) + \phi_1(x_1) u'(x_1),$$

which simplifies to

$$-\sum_{j=1}^N u_j \int_{x_1}^{x_2} \phi_1' \phi_j'(x) dx + \sum_{j=1}^N u_j \int_{x_1}^{x_2} p(x) \phi_1 \phi_j'(x) dx + \sum_{j=1}^N u_j \int_{x_1}^{x_2} q(x) \phi_1 \phi_j(x) dx = \alpha + \int_{x_1}^{x_2} \phi_1 f(x) dx. \quad (7)$$

The equation in the last interval as

$$-\sum_{j=1}^N u_j \int_{x_{N-1}}^{x_N} \phi_N' \phi_j'(x) dx + \sum_{j=1}^N u_j \int_{x_{N-1}}^{x_N} p(x) \phi_N \phi_j'(x) dx + \sum_{j=1}^N u_j \int_{x_{N-1}}^{x_N} q(x) \phi_N \phi_j(x) dx$$
$$= \int_{x_{N-1}}^{x_N} \phi_N f(x) dx - \phi_N(x_N) u'(x_N) + \phi_N(x_{N-1}) u'(x_{N-1}),$$

which simplifies to

$$-\sum_{j=1}^N u_j \int_{x_{N-1}}^{x_N} \phi_N' \phi_j'(x) dx + \sum_{j=1}^N u_j \int_{x_{N-1}}^{x_N} p(x) \phi_N \phi_j'(x) dx + \sum_{j=1}^N u_j \int_{x_{N-1}}^{x_N} q(x) \phi_N \phi_j(x) dx$$
$$= -\beta + \int_{x_{N-1}}^{x_N} \phi_N f(x) dx. \quad (8)$$

The equation of any interval in between the first and last, that is, for $i = 2, \ldots, N-1$, is given by

$$-\sum_{j=1}^N u_j \int_{x_{i-1}}^{x_{i+1}} \phi_i' \phi_j'(x) dx + \sum_{j=1}^N u_j \int_{x_{i-1}}^{x_{i+1}} p(x) \phi_i \phi_j'(x) dx + \sum_{j=1}^N u_j \int_{x_{i-1}}^{x_{i+1}} q(x) \phi_i \phi_j(x) dx = \int_{x_{i-1}}^{x_{i+1}} \phi_i f(x) dx. \quad (9)$$

Using the matrix-vector notation, this can be written as

$$\mathbf{Ku} = \mathbf{f},$$

where $\mathbf{K}$ is a tridiagonal matrix called the stiffness matrix, $\mathbf{u}$ is a column matrix for the unknowns, u_j, to be determined, and $\mathbf{f}$ is a column matrix for the integral of the forcing function and is called load vector. If $p(x)$ and $q(x)$ are expressions other than constants, then a suitable numerical integration technique can be used.

2.2 The nonlinear second-order BVP

We next formulate the case of nonlinear second-order two-point boundary value problems with an illustration based on Burgers' equation,

$$y''(x) + y(x)y'(x) + y(x) = f(x), \quad x \in [a, b], \tag{10}$$

subject to the boundary conditions $y'(a) = \alpha$ and $y'(b) = \beta$.

In this case, the formulation follows the same procedure as in the previous subsection, for the first and third term of the LHS of (10). Here, we pay attention to the nonlinear term $y(x)y'(x)$. Now, applying the Galerkin FEM to the nonlinear term, leads to

$$\int_a^b \phi_i u(x)u'(x)dx = \int_a^b \phi_i \left(\sum_{j=1}^{N} u_j\phi_j\right)\left(\sum_{j=1}^{N} u_j\phi'_j\right) dx \tag{11}$$

In the expansion of the terms in (11), it should be noted that the product of terms that involves nodes not adjacent to each other, is zero. That is, $u_j\phi_j u_k\phi'_k = 0$ for $k > j+1$. For example, the term. The product of terms that involves nodes adjacent to each other, are not zero and can easily be computed. That is, $u_j\phi_j u_k\phi'_k \neq 0$ for $k \leq j+1$.
In general,

$$\begin{aligned}
\int_a^b \phi_i u(x)u'(x)dx &= \int_a^b \phi_i(u_j\phi_j + u_{j+1}\phi_{j+1})(u_j\phi'_j + u_{j+1}\phi'_{j+1})dx \\
= u_j^2 \int_a^b \phi'_j\phi_i\phi_j dx &+ u_j u_{j+1}\int_a^b \phi'_{j+1}\phi_i\phi_j dx + u_{j+1}u_j \int_a^b \phi'_j\phi_i\phi_{j+1}dx + u_{j+1}^2 \int_a^b \phi'_{j+1}\phi_i\phi_{j+1}dx.
\end{aligned} \tag{12}$$

Now, setting $\phi_i = \phi_1, \ldots, \phi_i = \phi_N$, in turn, leads to a system of N nonlinear equations. The equation in the first interval appears as

$$u_1^2 \int_{x_1}^{x_2} \phi'_1\phi_1^2 dx + u_1u_2 \int_{x_1}^{x_2} \phi'_2\phi_1^2 dx + u_2u_1 \int_{x_1}^{x_2} \phi'_1\phi_1\phi_2 dx + u_2^2 \int_{x_1}^{x_2} \phi'_2\phi_1\phi_2 dx, \tag{13}$$

and the equation in the last interval is

$$u_{N-1}^2 \int_{x_{N-1}}^{x_N} \phi'_{N-1}\phi_{N-1}\phi_N dx + u_{N-1}u_N \int_{x_{N-1}}^{x_N} \phi'_N\phi_{N-1}\phi_N dx$$
$$+ u_N u_{N-1} \int_{x_{N-1}}^{x_N} \phi'_{N-1}\phi_N^2 dx + u_N^2 \int_{x_{N-1}}^{x_N} \phi'_N\phi_N^2 dx. \quad (14)$$

The equations in the intervals between the first and last, that is, for $i = 2, \ldots, N-1$, appear as

$$u_{i-1}^2 \int_{x_i}^{x_{i+1}} \phi'_{i-1}\phi_{i-1}\phi_i dx + u_{i-1}u_i \int_{x_i}^{x_{i+1}} \phi'_i\phi_{i-1}\phi_i dx + u_i u_{i-1} \int_{x_i}^{x_{i+1}} \phi'_{i-1}\phi_i^2 dx + u_i^2 \int_{x_i}^{x_{i+1}} \phi'_i\phi_i^2 dx$$
$$+ u_i u_{i+1} \int_{x_i}^{x_{i+1}} \phi'_{i+1}\phi_i^2 dx + u_{i+1}u_i \int_{x_i}^{x_{i+1}} \phi'_i\phi_i\phi_{i+1} dx + u_{i+1}^2 \int_{x_i}^{x_{i+1}} \phi'_{i+1}\phi_i\phi_{i+1} dx \quad (15)$$

When the values of the integrals in (13), (14), (15) and those of first and third terms of the LHS of (10) are computed, and the boundary conditions are applied, then the formulation of the problem (10) leads to the system of nonlinear equations,

$$\mathbf{F}(\mathbf{u}) = \mathbf{0},$$

where

$$\mathbf{F}(\mathbf{u}) = \begin{bmatrix} F_1(\mathbf{u}) \\ F_2(\mathbf{u}) \\ F_3(\mathbf{u}) \\ F_4(\mathbf{u}) \\ \vdots \\ F_N(\mathbf{u}) \end{bmatrix} = \begin{bmatrix} du_1 + bu_2 - \frac{1}{3}u_1^2 + \frac{1}{6}u_1u_2 + \frac{1}{6}u_2^2 - \int_{x_1}^{x_2} \phi_1 f(x)dx + \alpha \\ bu_1 + 2du_2 + bu_3 - \frac{1}{6}u_1^2 - \frac{1}{6}u_1u_2 + \frac{1}{6}u_2u_3 + \frac{1}{6}u_3^2 - \int_{x_1}^{x_3} \phi_2 f(x)dx \\ bu_2 + 2du_3 + bu_4 - \frac{1}{6}u_2^2 - \frac{1}{6}u_2u_3 + \frac{1}{6}u_3u_4 + \frac{1}{6}u_4^2 - \int_{x_2}^{x_4} \phi_3 f(x)dx \\ bu_3 + 2du_4 + bu_5 - \frac{1}{6}u_3^2 - \frac{1}{6}u_3u_4 + \frac{1}{6}u_4u_5 + \frac{1}{6}u_5^2 - \int_{x_3}^{x_5} \phi_4 f(x)dx \\ \vdots \\ bu_{N-1} + du_N - \frac{1}{6}u_{N-1}^2 - \frac{1}{6}u_{N-1}u_N + \frac{1}{3}u_N^2 - \int_{x_{n-1}}^{x_N} \phi_N f(x)dx - \beta \end{bmatrix}, \quad (16)$$

with $d = h/3 - 1/h$ and $b = h/6 + 1/h$.

For number of nodes, N, one may write (16) by separating it into three parts, the linear, nonlinear, and the integral involving $f(x)$and boundary conditions. In general,

$$\mathbf{F}_{linear} = \begin{cases} du_i + bu_{i+1} & \text{for } i = 1, \\ bu_{i-1} + 2du_i + bu_{i+1} & \text{for } i = 2, \ldots, N-1, \\ bu_{i-1} + du_i & \text{for } i = N. \end{cases}$$

$$\mathbf{F}_{nonlinear} = \begin{cases} -\frac{1}{3}u_i^2 + \frac{1}{6}u_i u_{i+1} + \frac{1}{6}u_{i+1}^2 & \text{for } i = 1, \\ -\frac{1}{6}u_{i-1}^2 - \frac{1}{6}u_{i-1}u_i + \frac{1}{6}u_i u_{i+1} + \frac{1}{6}u_{i+1}^2 & \text{for } i = 2, \dots, N-1, \\ -\frac{1}{6}u_{i-1}^2 - \frac{1}{6}u_{i-1}u_i + \frac{1}{3}u_i^2 & \text{for } i = N. \end{cases}$$

$$\mathbf{F}_{phi} = \begin{cases} \int\limits_{x_i}^{x_{i+1}} \phi_i f(x)dx - \alpha & \text{for } i = 1, \\ \int\limits_{x_{i-1}}^{x_{i+1}} \phi_i f(x)dx & \text{for } i = 2, \dots, N-1, \\ \int\limits_{x_{i-1}}^{x_i} \phi_i f(x)dx + \beta & \text{for } i = N. \end{cases}$$

The system,

$$\mathbf{F}(\mathbf{u}) = \mathbf{F}_{linear} + \mathbf{F}_{nonlinear} - \mathbf{F}_{phi} = \mathbf{0},$$

is then solved using Newton's method, among other numerical techniques. This method required one to solve the nonlinear system of equations $\mathbf{F}(\mathbf{u}) = \mathbf{0}$ by finding the inverse of the Jacobian matrix, $\mathbf{J(u)}$, and the iterative solution is obtained using

$$\mathbf{u}^{(n+1)} = \mathbf{u}^{(n)} - \mathbf{J^{-1}}(\mathbf{u}^{(n)})\mathbf{F}(\mathbf{u}^{(n)}),$$

where

$$\mathbf{J}(\mathbf{u}) = \begin{bmatrix} \frac{\partial F_1}{\partial u_1} & \frac{\partial F_1}{\partial u_2} & \cdots & \frac{\partial F_1}{\partial u_N} \\ \frac{\partial F_2}{\partial u_1} & \frac{\partial F_2}{\partial u_2} & \cdots & \frac{\partial F_2}{\partial u_N} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial F_N}{\partial u_1} & \frac{\partial F_N}{\partial u_2} & \cdots & \frac{\partial F_N}{\partial u_N} \end{bmatrix}$$

When $||\mathbf{u}^{(n+1)} - \mathbf{u}^{(n)}|| = \mathbf{0}$, then the system $\mathbf{F}(\mathbf{x}) = \mathbf{0}$ has converged to the solution.
In this case, the $\mathbf{J(u)}$ came up as a tridiagonal matrix, whose diagonal elements are given as

$$\mathbf{J}(i,i) = \begin{cases} d - \frac{2}{3}u_i + \frac{1}{6}u_{i+1}, & \text{for } i = 1, \\ 2d - \frac{1}{6}u_{i-1} + \frac{1}{6}u_{i+1} & \text{for } i = 2, \dots, N-1, \\ d - \frac{1}{6}u_{i-1} + \frac{2}{3}u_i & \text{for } i = N. \end{cases}$$

and for the entries above and below the diagonal, respectively, for $i = 1, \dots, N-1$,

$$\mathbf{J}(i, i+1) = b + \frac{1}{6}u_i + \frac{1}{3}u_{i+1},$$

and

$$\mathbf{J}(i+1, i) = b - \frac{1}{3}u_i - \frac{1}{6}u_{i+1}.$$

3 Test problems

In this section, three problems of second order two point BVPs with Neumann boundary conditions with different parameters were solved. Mathematical simulations were done with the help of MATLAB software.

3.1 Problem 1

Consider the following second-order linear two-point boundary-value problem with constant coefficients

$$y''(x) + y(x) + x = 0, \quad x \in [0,1], \tag{17}$$

with Neumann boundary conditions $y'(0) = -1 + \csc(1)$ and $y'(1) = -1 + \cot(1)$. The exact solution is

$$y(x) = -x + \csc(1)\sin(x).$$

The approximations are compared with the exact solutions for problem 1 at the selected nodes as shown in Table 1. The errors corresponding to each of the approximations are shown in Table 2. The approximations and the errors have been illustrated in Figures 1 and 2. Graphs for the comparison between the approximation and the exact solution for the different number of nodes, are shown in Figure 3.

Table 1: The comparison of approximation with exact solution for problem 1 for different number of nodes N.

	Exact	Number of nodes					
x		5	9	17	33	65	129
0.00	0.00000000	0.00308378	0.00077300	0.00019338	0.00004835	0.00001209	0.00000302
0.25	0.04401365	0.04702430	0.04476705	0.04420205	0.04406076	0.04402543	0.04401660
0.50	0.06974696	0.07259219	0.07045760	0.06992458	0.06979137	0.06975806	0.06974974
0.75	0.06005617	0.06274201	0.06072571	0.06022343	0.06009798	0.06006662	0.06005878
1.00	0.00000000	0.00261913	0.00065198	0.00016282	0.00004069	0.00001017	0.00000254

Table 2: Errors in the approximations to the linear problem with constant coefficients for different number of nodes, N.

	Number of nodes					
x	5	9	17	33	65	129
0.00	0.00308378	0.00077300	0.00019338	0.00004835	0.00001209	0.00000302
0.25	0.00301065	0.00075340	0.00018840	0.00004710	0.00001178	0.00000294
0.50	0.00284523	0.00071064	0.00017762	0.00004440	0.00001110	0.00000278
0.75	0.00268584	0.00066954	0.00016727	0.00004181	0.00001045	0.00000261
1.00	0.00261913	0.00065198	0.00016282	0.00004069	0.00001017	0.00000254

3.2 Problem 2

Consider the second-order linear boundary value problem with non-constant coefficients,

$$y''(x) = -2(1 - 2x^2)y(x) \tag{18}$$

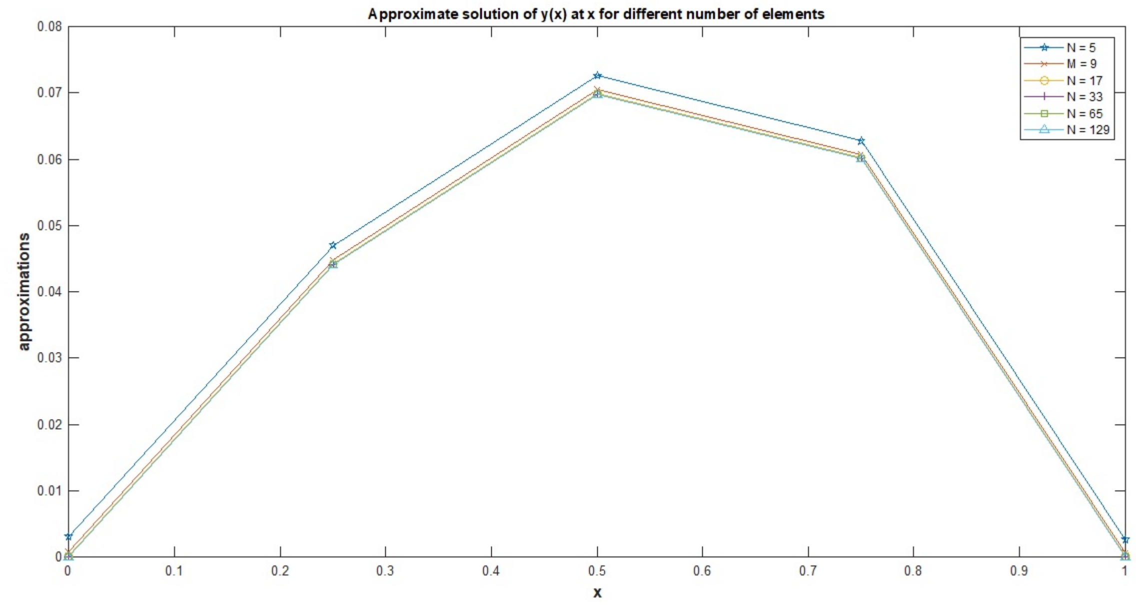

Figure 1: Approximations to the linear problem for different element sizes h.

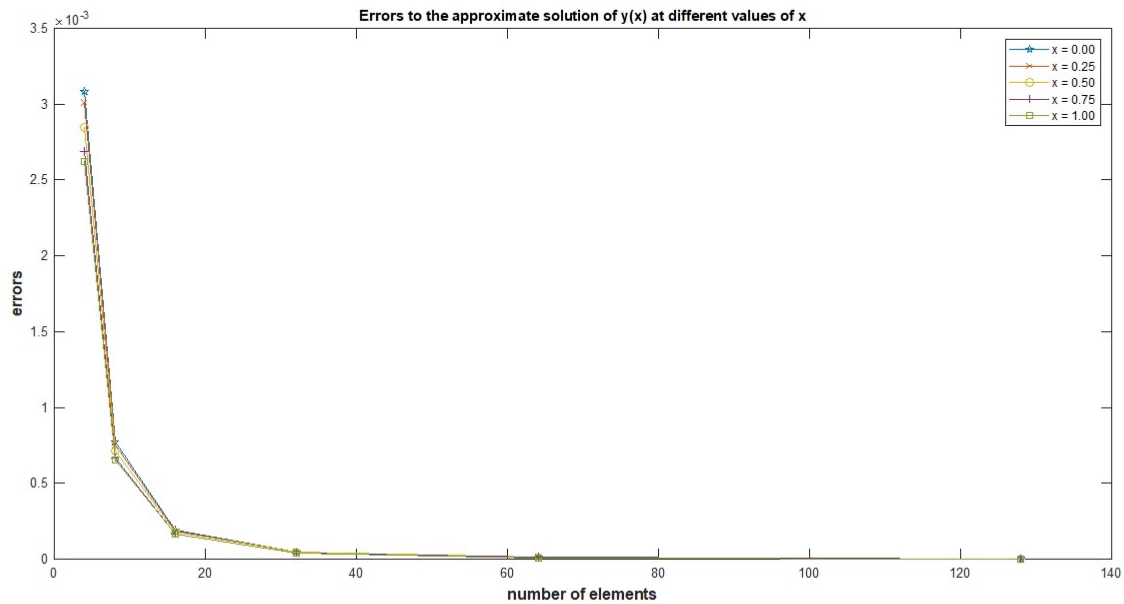

Figure 2: Errors in the approximations to the linear problem for different element sizes h.

over $[0, 1]$ subject to the boundary conditions $y'(0) = 1$ and $y'(1) = -2/\exp(1)$. The known exact solution to this problem is

$$y(x) = \exp(-x^2).$$

The approximations are compared with the exact solutions for problem 2 at the selected nodes as shown in Table 3. The errors corresponding to each of the approximations are shown in Table 4. The approximations and the errors have been illustrated in Figures 4

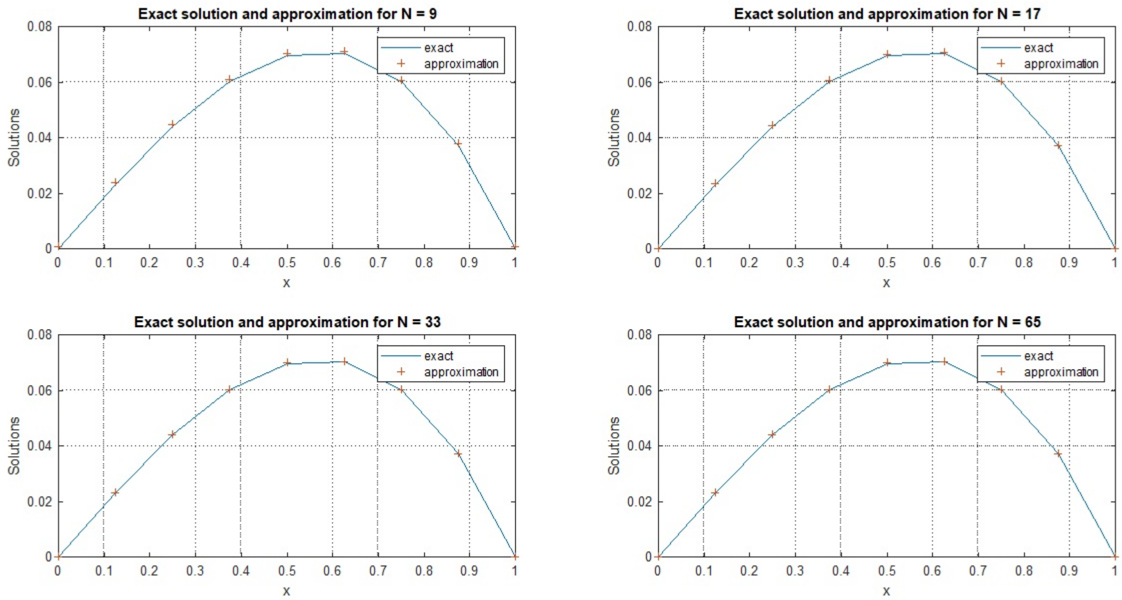

Figure 3: Comparison between the approximations and exact solution of the linear problem, for different number of nodes, N.

and 5. Graphs for the comparison between the approximation and the exact solution for the different step sizes, are shown in Figure 6.

Table 3: The comparison of numerical solution with exact solution for problem 2 with different number of nodes, N.

x	Exact	Number of nodes 5	9	17	33	65	129
0.00	1.00000000	1.01155608	1.00292324	1.00073285	1.00018334	1.00004584	1.00001146
0.25	0.93941306	0.95086788	0.94231146	0.94013973	0.93959486	0.93945852	0.93942443
0.50	0.77880079	0.78982824	0.78159159	0.77950050	0.77897584	0.77884455	0.77881173
0.75	0.56978282	0.58001957	0.57237142	0.57043172	0.56994516	0.56982341	0.56979297
1.00	0.36787944	0.37758854	0.37032930	0.36849327	0.36803298	0.36791783	0.36788904

3.3 Problem 3

In this case, we considered the Burgers' equation

$$y''(x) + y(x)y'(x) + y(x) = \frac{1}{2}\sin(2x), \quad x \in [0, \pi/2], \tag{19}$$

subject to the boundary conditions $y'(0) = 1$ and $y'(\pi/2) = 0$. This is a nonlinear boundary-value problem, whose known exact solution is

$$y(x) = \sin(x).$$

Table 4: Errors in the approximations to the linear problem with non-constant coefficients for different element sizes, h.

x	Number of nodes 5	9	17	33	65	129
0.00	0.01155608	0.00292324	0.00073285	0.00018334	0.00004584	0.00001146
0.25	0.01145482	0.00289839	0.00072667	0.00018179	0.00004546	0.00001136
0.50	0.01102745	0.00279081	0.00069972	0.00017506	0.00004377	0.00001094
0.75	0.01023675	0.00258859	0.00064890	0.00016233	0.00004059	0.00001015
1.00	0.00970910	0.00244986	0.00061383	0.00015354	0.00003839	0.00000960

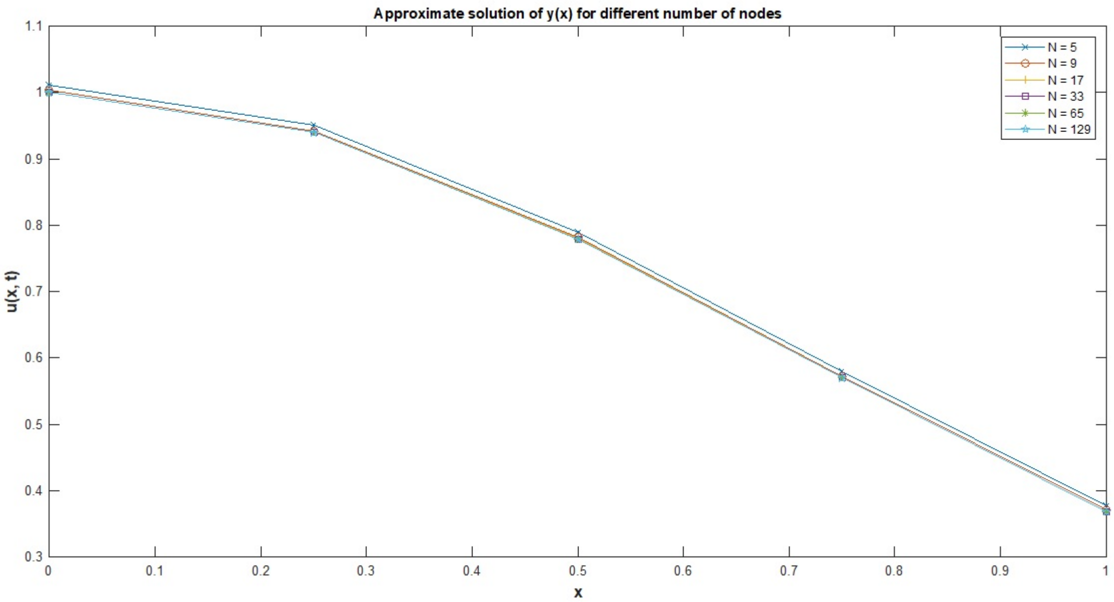

Figure 4: Approximations to the linear problem for different element sizes h.

The system of nonlinear equations obtained in problem 3 are solved using Newton's method starting with $\mathbf{u} = (0, 0, 0, 0, 0)$ to generate the approximations and with five iterations, the solution was obtained as shown in (5).

The approximations at the selected nodes, all for five iterations are compared with the exact solutions for problem 3 as shown in Table 6. The errors corresponding to each of the approximations are shown in Table 7. The approximations and the errors have been illustrated in Figures 7 and 8. Graphs for the comparison between the approximation and the exact solution for the different number of nodes, are shown in Figure 9

4 Discussion of results

Convergence means that the solution to the Galerkin FEM approximation approaches the true solution of the two-point BVP of ODE when the mesh is refined or step size, h goes

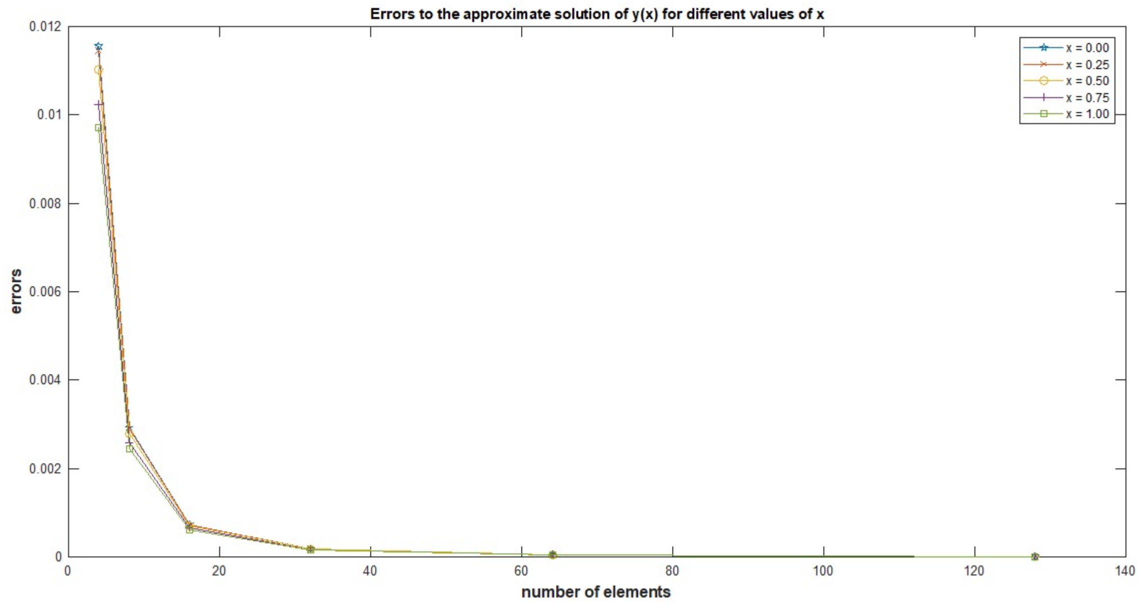

Figure 5: Errors in the approximations to the linear problem with constant coefficients for different element sizes h.

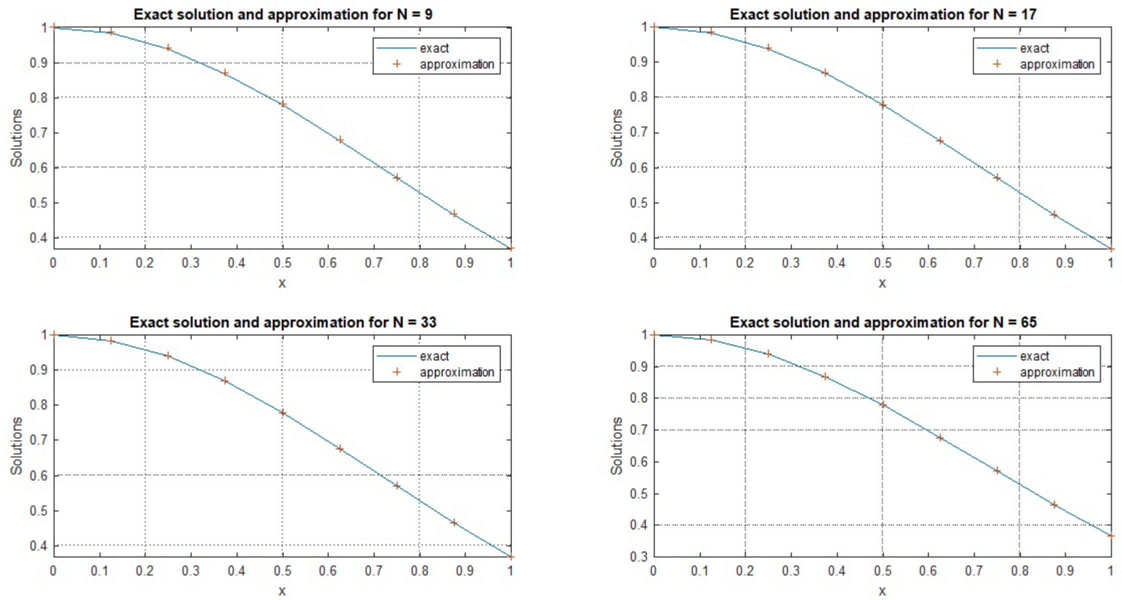

Figure 6: Comparison between the approximations and exact solution of the linear problem with non constant coefficients, for different number of nodes, N.

to zero. As a test for convergence, it is required to determine whether the solutions with increasingly small step sizes approach the exact solution. Additionally, if the analytical solution is known, one tests whether the sequence of approximate solutions as step sizes reduce, converges to a fixed value. This is known as consistency. On the other hand, rate

Table 5: Approximations to the nonlinear problem for $h = \pi/8$, for 5 iterations.

x	Iterations 1	2	3	4	5
0	0.34311619	0.00243246	-0.00530156	-0.00531513	-0.00531513
$\pi/8$	0.70972273	0.38419793	0.37848506	0.37847603	0.37847603
$\pi/4$	1.02010293	0.70767026	0.70677010	0.70677297	0.70677297
$3\pi/8$	1.24850085	0.92615224	0.92936327	0.92937645	0.92937645
$\pi/2$	1.33966807	1.00674181	1.01127053	1.01128679	1.01128679

Table 6: The comparison of approximations with exact solutions for problem 3 with different element sizes h, up to 5 iterations.

x	Exact	Element sizes					
		$h = \pi/8$	$h = \pi/16$	$h = \pi/32$	$h = \pi/64$	$h = \pi/128$	$h = \pi/256$
0	0.00000000	-0.00531513	-0.00132187	-0.00033005	-0.00008249	-0.00002062	-0.00000515
$\pi/8$	0.38268344	0.37847603	0.38164356	0.38242421	0.38261867	0.38266725	0.38267939
$\pi/4$	0.70710678	0.70677297	0.70704664	0.70709320	0.70710348	0.70710596	0.70710658
$\frac{3\pi}{8}$	0.92387953	0.92937645	0.92527753	0.92423049	0.92396736	0.92390150	0.92388502
$\pi/2$	1.00000000	1.01128679	1.00283009	1.00070802	1.00017704	1.00004426	1.00001107

Table 7: Errors in the approximations to the nonlinear problem for different element sizes h, up to 5 iterations.

x	Element sizes					
	$h = \pi/8$	$h = \pi/16$	$h = \pi/32$	$h = \pi/64$	$h = \pi/128$	$h = \pi/256$
0	0.00531513	0.00132187	0.00033005	0.00008249	0.00002062	0.00000515
$\pi/8$	0.00420741	0.00103988	0.00025922	0.00006476	0.00001619	0.00000405
$\pi/4$	0.00033381	0.00006014	0.00001358	0.00000330	0.00000082	0.00000020
$3\pi/8$	0.00549692	0.00139799	0.00035096	0.00008783	0.00002196	0.00000549
$\pi/2$	0.01128679	0.00283009	0.00070802	0.00017704	0.00004426	0.00001107

of convergence is a measure of how fast or how slow the finite element solution converges to the exact solution of a given problem. In other words, the rate of convergence is how fast the error tends to zero with the mesh-size, h [17]. The order of convergence estimates the actual rate of convergence or the speed at which the errors go to zero. Typically, the order estimates this rate in terms of polynomial behavior.

Computations of approximations using Galerkin FEM and errors have been made for three test problems, with different number of nodes, and results tabulated. Errors have been computed at selected nodes for different step sizes.

In Test Problem 1, the linear problem with constant coefficients is solved using Galerkin

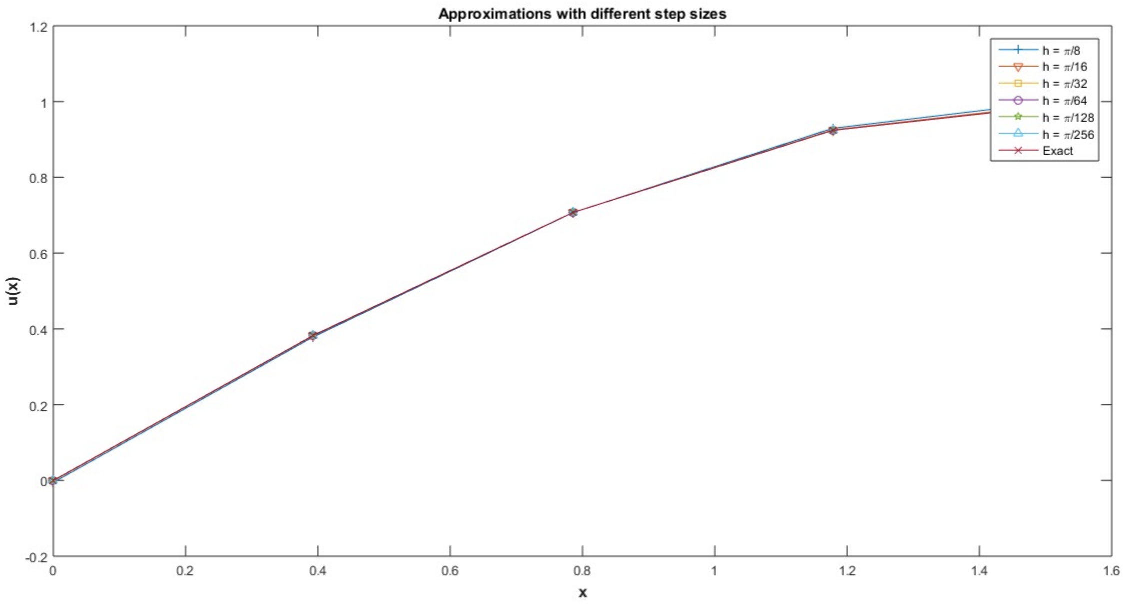

Figure 7: Approximations to the nonlinear problem for different element sizes h, up to 5 iterations.

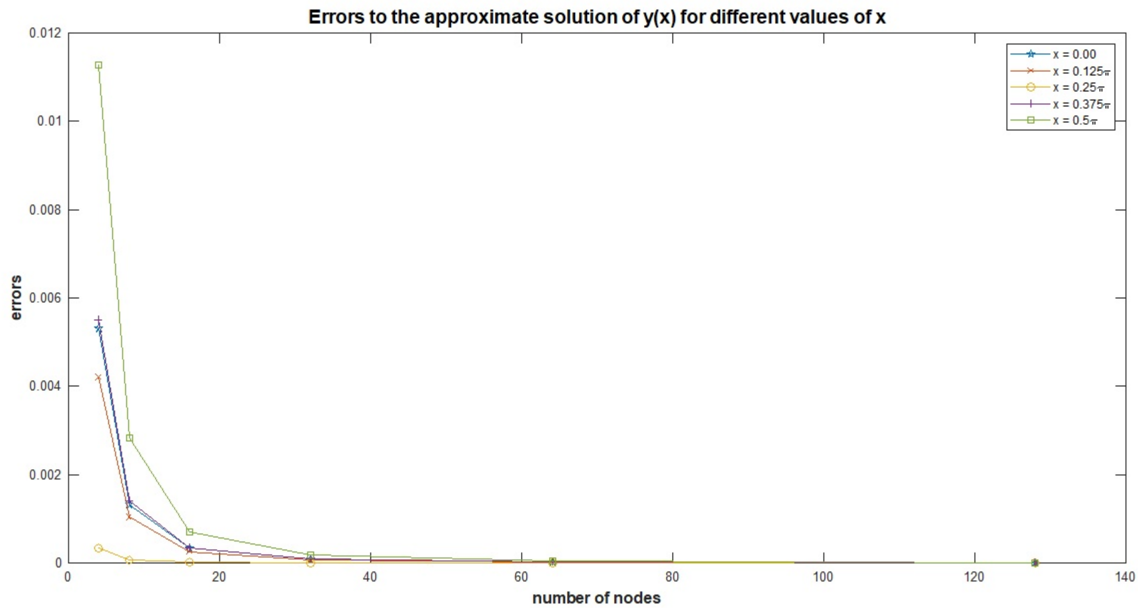

Figure 8: Errors in the approximations to the nonlinear problem for different element sizes h, up to 5 iterations.

FEM with different number of nodes, N. Table 1 shows the numerical solutions at selected points for different number of nodes and Table 2 shows the errors at selected points for selected number of nodes. Figure 1 shows the approximations obtained at selected points by reducing step sizes. The graphs in Figure 3 show the comparison between the analytical and

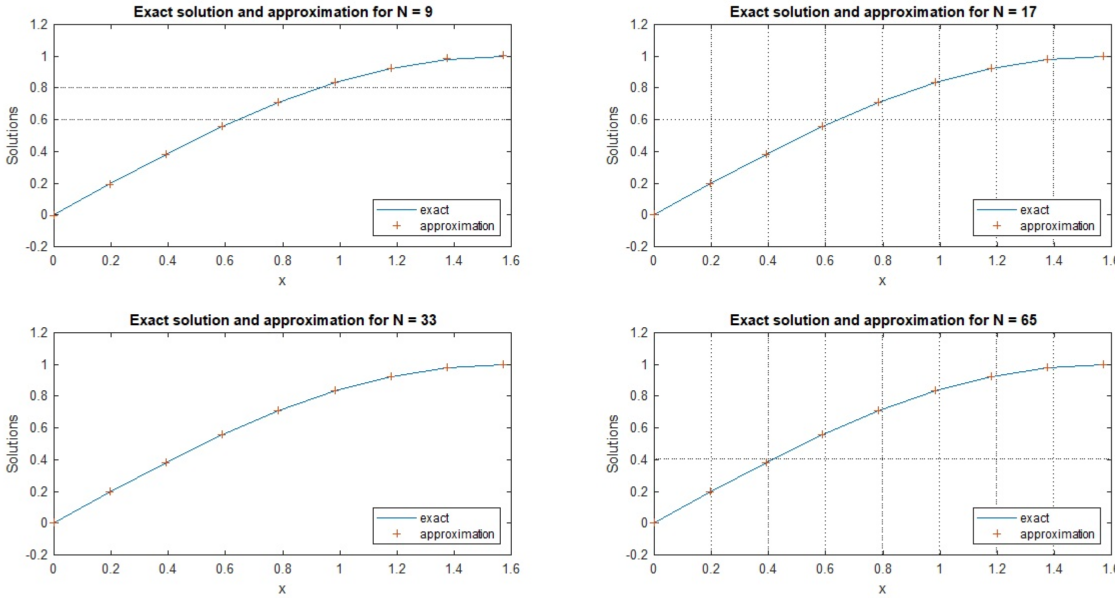

Figure 9: Comparison between the approximations and exact solution of the nonlinear problem, for different number of nodes, N.

approximate solutions for different number of elements. The results indicated that Galerkin FEM is convergent because reductions in step sizes led to little change in the approximations. Also, the variation of errors with step sizes is shown in Figure 2, indicating that errors reduce with reduction in step sizes. The rate of convergence was calculated, and it was found to have order of convergence of 2, called quadratic convergence.

Galerkin FEM was applied to the Test Problem 2 which has non-constant coefficients, using different number of nodes, N. Table 3 gives approximations as step sizes are reduced, showing that the approximate solution continuously improves as the step size reduces, and in the end the values are within the same range. From the results obtained in Table 4, it is observed that the errors at selected points reduce as step sizes reduce. Thus, the consistency and convergence of Galerkin FEM is attained since subsequent reductions in step sizes lead to small changes in the approximations. In Figure 4, the graphs show that the approximations obtained at selected points by reducing step sizes are very close to each other. In Figure 5, the graphs of errors are initially distinguished, though they tend to one point as the step size reduces, indicating that a point of convergence has been attained. From Figure 6, the graphs of exact and approximations for different number of nodes clearly show that approximate solution becomes exact as element size goes to zero. The rate of convergence was calculated and the order of convergence was found to be 2, which is a quadratic convergence.

In Test Problem 3, the nonlinear problem is solved using Galerkin FEM with different step sizes. Table 5 shows that the approximate solution for $h = \pi/8$ was obtained with only 5 iterations. Table 6 shows that approximations tended to a fixed value, which is the exact solution as step sizes reduce. Table 7 shows the errors at selected points reducing as step sizes decrease. The Figure 7 shows the results obtained at selected points for reducing step sizes. In Figure 8, the graphs of errors are initially distinguished, though they tend to one point as the step size reduces, indicating that a point of convergence has been attained. From Figure

9, the graphs of exact and approximations for different number of elements clearly shows that approximate solution becomes exact as element size goes to zero. The results indicate that Galerkin FEM is consistent and convergent because reductions in step sizes lead to little change in the approximation results. Also, the rate of convergence was calculated, and it was found to have order of convergence of 2, referred to as quadratic convergence.

5 Conclusion

In this study, Galerkin FEM has been developed to approximate the solution of both second-order linear with constant and non-constant coefficients, and nonlinear second-order two-point BVP of ordinary differential equations with Neumann boundary conditions. Lagrange linear piece-wise polynomials have been used as trial functions.

Linear second order two-point BVP of ODEs with non-constant coefficient was solved by applying Gauss quadrature 3-point rule in the Galerkin FEM. For the nonlinear BVP, the Newton's method was used with the Galerkin FEM. The errors in approximations have been studied, noting that for this method, errors in the approximations reduce with decreasing element or step size. The convergence and consistency of Galerkin FEM applied to the linear and nonlinear second-order boundary value problems of ordinary differential equations have been discussed. Basing on the results from the simulations, it was found that the method considered was stable. Again, for this method, it was noted that the method is convergent and consistent since further reduction of element or step sizes produced insignificant reduction in the error of all test problems. Also, the rate of convergence of proposed method on all the three test problems was found to converge with order 2. This means that the proposed method has a quadratic convergence. Thus,the method developed performs well with both linear and nonlinear two point BVPs. The results have been presented in a number of tables and illustrated using graphs, all generated using MATLAB.

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