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COUNTING NEGATIVE EIGENVALUES OF ONE-DIMENSIONAL SCHRÖDINGER OPERATORS WITH SINGULAR POTENTIALS

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ABSTRACT. In this paper, we extend the well known estimates for the number of negative eigenvalues of one-dimensional Schrödinger operators with potentials that are absolutely continuous with respect to the Lebesgue measure to the case of strongly singular potentials.

1. INTRODUCTION

Let $N_-(V)$ be the number of negative eigenvalues of a Schrödinger operator

$$H = -\Delta - V, \ V \geq 0$$

on $L^2(\mathbb{R}^d)$. For $d > 2$, the number $N_-(V)$ is estimated above by the well known Cwikel-Lieb-Rozenblum (CLR) inequality [2, 8]. For $d = 2$, the CLR inequality fails and the best known estimates for $N_-(V)$ in this case involve weighted L^1 norms and Orlciz norms of the potential (see, e.g., [9, 10] and [7] in the case where V is supported by a Lipschitz curve). For $d = 1$, an analogue of the CLR inequality holds for potentials that are monotone on $\mathbb{R}_+$ and $\mathbb{R}_-$ (see, e.g., [4]). For general nonnegative potentials that are locally integrable on $\mathbb{R}$ with respect to the standard Lebesgue measure, $N_-(V)$ admits the following estimate

$$N_-(V) \leq 1 + C \sum_{\{j\in\mathbb{Z},\, \mathcal{A}_j(V)>c\}} \sqrt{\mathcal{A}_j(V)}\,, \tag{1.1}$$

where C, c are positive constants and

$$\mathcal{A}_0(V) = \int_{-1}^{1} V(t)\,dt, \ \ \mathcal{A}_j(V) = 2^j \int_{2^{j-1}}^{2^j} V(t)\,dt, \ j > 0,$$

$$\mathcal{A}_j(V) = 2^{|j|} \int_{-2^{|j|}}^{-2^{|j|-1}} V(t)\,dt, \ j < 0$$

(see [11] and the references therein). When V is a linear combination of Dirac delta functions, results on $N_-(V)$ can be found for example in [1]. The main purpose of this paper is to extend the estimate (1.1) to the case when V is allowed

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to be a measure that is not necessarily absolutely continuous with respect to the Lebesgue measure. In particular, we study the operator

$$H_\mu := -\frac{d^2}{dx^2} - \mu \tag{1.2}$$

on $L^2(\mathbb{R})$, where μ is an arbitrary σ-finite positive Radon measure on $\mathbb{R}$.

2. Main result

We denote by $N_-(\mu, \mathbb{R})$ the number of negative eigenvalues of (1.2) counting multiplicities. Define (1.2) via its quadratic form

$$q_{\mu,\mathbb{R}}[u] := \int_{\mathbb{R}} |u'(x)|^2 dx - \int_{\mathbb{R}} |u(x)|^2 d\mu(x),$$
$$\mathrm{Dom}(q_{\mu,\mathbb{R}}) = W_2^1(\mathbb{R}) \cap L^2(\mathbb{R}, d\mu),$$

where $W_2^1(\mathbb{R})$ denotes the standard Sobolev space of square integrable functions with square integrable weak derivatives. Then $N_-(\mu, \mathbb{R})$ is given by

$$N_-(\mu, \mathbb{R}) = \sup\{\dim L \,:\, q_{\mu,\mathbb{R}}[u] < 0,\ \forall u \in L \setminus \{0\}\}, \tag{2.1}$$

where L denotes a linear space of $\mathrm{Dom}(q_{\mu,\mathbb{R}})$ (see, e.g., [3, Theorem 10.2.3]).

Let

$$I_n := [2^{n-1}, 2^n],\ n > 0, \quad I_0 := [-1, 1], \quad I_n := [-2^{|n|}, -2^{|n|-1}],\ n < 0$$

and

$$\mathcal{A}_n := \int_{I_n} |x|\, d\mu(x) \ \ n \neq 0\,,\ \mathcal{A}_0 := \int_{I_0} d\mu(x)\,. \tag{2.2}$$

Theorem 2.1. *Let μ be a σ-finite positive Radon measure on $\mathbb{R}$ and let $\{\mathcal{A}_n\}$ be the sequence in (2.2). Then there exist constants $c, C > 0$ such that*

$$N_-(\mu, \mathbb{R}) \le 1 + C \sum_{\{n\in\mathbb{Z},\, \mathcal{A}_n > c\}} \sqrt{\mathcal{A}_n}\,.$$

3. Auxiliary results

Let $\Omega \subset \mathbb{R}^n$ be an arbitrary open set and let μ be a positive σ-finite Radon measure on $\mathbb{R}^n$. Further, let V be a non-negative μ-measurable real valued function and $V \in L^1_{\mathrm{loc}}(\overline{\Omega}, \mu)$. Define the following quadratic form

$$\mathcal{E}_{V\mu,\Omega}[w] := \int_\Omega |\nabla w|^2\, dx - \int_{\overline{\Omega}} V|w|^2\, d\mu(x),$$

with the domain $\mathrm{Dom}(\mathcal{E}_{V\mu,\Omega})$, which is a linear subspace of $W_2^1(\Omega) \cap L^2(\overline{\Omega}, V d\mu)$. Note that μ does not have to be the n dimensional Lebesgue measure, and it may well happen that $\mu(\partial\Omega) > 0$.

Definition 3.1. Let $\Omega \subset \mathbb{R}^n$ be an open set. We say that a (finite or infinite) sequence $\{\Omega_k\}$ of non-empty open subsets $\Omega_k \subset \Omega$ is a μ–partition of Ω if $\Omega_k \cap \Omega_l = \emptyset$ when $k \neq l$, $\ \Omega \setminus \cup_k \Omega_k$ has zero Lebesgue measure, and $\mu\left(\overline{\Omega} \setminus \cup_k \overline{\Omega}_k\right) = 0$.

The following result can be found, e.g., in [5, Ch.6, §2.1, Theorem 4] in the case when μ is absolutely continuous with respect to the Lebesgue measure.

Lemma 3.2. *Let $\{\Omega_k\}$ be a μ–partition of Ω and suppose* $\mathrm{Dom}(\mathcal{E}_{V\mu,\Omega})$, $\mathrm{Dom}(\mathcal{E}_{V\mu,\Omega_k})$ *are such that for every k,*

$$w|_{\Omega_k} \in \mathrm{Dom}(\mathcal{E}_{V\mu,\Omega_k}), \quad \forall w \in \mathrm{Dom}(\mathcal{E}_{V\mu,\Omega}).$$

Then

$$N_-(\mathcal{E}_{V\mu,\Omega}) \le \sum_k N_-(\mathcal{E}_{V\mu,\Omega_k}). \tag{3.1}$$

Proof. Let

$$\Sigma := \oplus\{\mathrm{Dom}(\mathcal{E}_{V\mu,\Omega_k}),\ k = 1, 2, ...\}.$$

Here $\oplus$ denotes the direct sum. We consider $\sum_k \mathcal{E}_{V\mu,\Omega_k}$ as a form defined on Σ. Let $\mathcal{J} : \mathrm{Dom}(\mathcal{E}_{V\mu,\Omega}) \longrightarrow \Sigma$ be the embedding defined by

$$w \longmapsto (w|_{\Omega_1}, w|_{\Omega_2}, ...).$$

Let $\Gamma := \mathcal{J}(\mathrm{Dom}(\mathcal{E}_{V\mu,\Omega}))$. Then $\forall w \in \mathrm{Dom}(\mathcal{E}_{V\mu,\Omega})$, we have

$$\begin{aligned}
\mathcal{E}_{V\mu,\Omega}[w] &= \int_\Omega |\nabla w(x)|^2 dx - \int_{\overline{\Omega}} V(x)|w(x)|^2 d\mu(x) \\
&\ge \sum_k \left(\int_{\Omega_k} |\nabla w(x)|^2 dx - \int_{\overline{\Omega}_k} V(x)|w(x)|^2 d\mu(x)\right) \\
&= \sum_k \mathcal{E}_{V\mu,\Omega_k}[w|_{\Omega_k}] = \left(\sum_k \mathcal{E}_{V\mu,\Omega_k}\right)[\mathcal{J}w].
\end{aligned}$$

Hence

$$N_-(\mathcal{E}_{V\mu,\Omega}) \le N_-\left(\left(\sum_k \mathcal{E}_{V\mu,\Omega_k}\right)\Big|_\Gamma\right) \le N_-\left(\sum_k \mathcal{E}_{V\mu,\Omega_k}\right) = \sum_k N_-(\mathcal{E}_{V\mu,\Omega_k}).$$

□

Let I be a bounded interval in $\mathbb{R}$ of length l. For simplicity, take $I = (0, l)$. Let $0 = t_0 < t_1 < ... < t_n = l$ be a partition of the interval I into n subintervals $I_k = (t_{k-1}, t_k)$. Let P stand for any such partition and $|P|$ denote the number of subintervals, i.e. $|P| = n$. Let ν be a positive Radon measure on $\mathbb{R}$ and for any real number $a > 0$, consider the following function of partitions:

$$\Theta_a(P) := \max_k (t_k - t_{k-1})^a\, \nu\left(\overline{I_k}\right). \tag{3.2}$$

Lemma 3.3. *Suppose $\nu(\{x\}) = 0$ for all $x \in \overline{I}$. Then for any $n \in \mathbb{N}$, there exists a partition P of the interval I such that $|P| = n$ and*

$$\Theta_a(P) \le l^a n^{-1-a} \nu(I). \tag{3.3}$$

Proof. The proof is similar to that of [11, Lemma 7.1] where measures absolutely continuous with respect to the Lesbegue measure were considered. By scaling, it is enough to prove (3.3) for $l = 1$ and $\nu(I) = 1$. For $n = 1$, there is nothing to prove. Now suppose (3.3) is true for some n. We need to show that then this is true for $n+1$. Since $x \longmapsto \nu([x,1))$ is continuous, there exists a point $x \in (0,1)$ such that

$$(1-x)^a \nu([x,1)) = (n+1)^{-1-a}. \tag{3.4}$$

Then one has

$$\nu([x,1)) = (n+1)^{-1-a}(1-x)^{-a}.$$

By the induction assumption, there exists a partition P_0 of the interval $(0,x)$ into n subintervals $0 = t_0 < t_1 < ... < t_n = x$ such that

$$\begin{aligned} \Theta_a(P_0) &\le x^a n^{-1-a}\nu((0,x)) \\ &= x^a n^{-1-a}\left(1-(n+1)^{-1-a}(1-x)^{-a}\right). \end{aligned}$$

Let P be the partition $0 = t_0 < t_1 < ... < t_n < t_{n+1} = 1$. Since (3.4) holds, (3.3) with $n+1$ in place of n will follow if one proves that $\Theta_a(P_0) \le (n+1)^{-1-a}$. The latter is achieved this by showing that

$$n^{-1-a} \le (n+1)^{-1-a}x^{-a} + n^{-1-a}(n+1)^{-1-a}(1-x)^{-a}.$$

Let $h(x) = (n+1)^{-1-a}x^{-a} + n^{-1-a}(n+1)^{-1-a}(1-x)^{-a}$. Then h is convex on $(0,1)$, and solving $h'(x) = 0$ we see that h attains its minimum on $(0,1)$ at the point $x = n(n+1)^{-1}$ and that this minimum value is n^{-1-a}. □

Lemma 3.4. *Suppose $\nu(\{t\}) = 0$ for all $t \in \overline{I}$. For any $n \in \mathbb{N}$, there exists a partition P of the interval I such that $|P| = n$ and*

$$\int_I |u(t)|^2 d\nu(t) \le \frac{l}{n^2}\nu(I)\int_I |u'(t)|^2\,dt$$

for all $u \in \mathcal{L}_n$, where $\mathcal{L}_n$ is the subspace of $W_2^1(I)$ of co-dimension n formed by the functions satisfying $u(t_1) = ... = u(t_n) = 0$.

Proof. For any $t \in I_k$, the Cauchy-Schwartz inequality implies

$$\begin{aligned} |u(t)|^2 &= |u(t)-u(t_k)|^2 = \left|\int_t^{t_k} u'(s)\,ds\right|^2 \le |t-t_k|\int_t^{t_k} |u'(s)|^2\,ds \\ &\le |t_k - t_{k-1}|\int_{t_{k-1}}^{t_k} |u'(s)|^2\,ds. \end{aligned}$$

Hence

$$\begin{aligned} \int_{I_k} |u(t)|^2\,d\nu(t) &\le \sup_{t\in I_k}|u(t)|^2\nu(I_k) \\ &\le |t_k - t_{k-1}|\nu(I_k)\int_{t_{k-1}}^{t_k} |u'(s)|^2\,ds. \end{aligned}$$

With $a = 1$, (3.2) and Lemma 3.3 imply

$$\begin{aligned}\int_I |u(t)|^2\, d\nu(t) &= \sum_{k=1}^{n} \int_{I_k} |u(t)|^2\, d\nu(t) \\ &\le \sum_{k=1}^{n} |t_k - t_{k-1}|\nu(I_k) \int_{t_{k-1}}^{t_k} |u'(s)|^2\, ds \\ &\le \Theta_a(P) \sum_{k=1}^{n} \int_{I_k} |u'(s)|^2\, ds \le \frac{l}{n^2}\nu(I) \int_I |u'(s)|^2\, ds.\end{aligned}$$

□

The above Lemma excludes measures with atoms. However, one can show that the lemma still holds true even when ν has atoms by approximating ν by measures that are absolutely continuous with respect to the Lebesgue measure.

Lemma 3.5. *Let ν be an arbitrary positive Radon measure on $\mathbb{R}$. For any $c > 1$ and any $n \in \mathbb{N}$ there exists a partition P of I such that $|P| = n$ and*

$$\int_{\overline{I}} |u(t)|^2 d\nu(t) \le c\,\frac{l}{n^2}\nu\left(\overline{I}\right) \int_I |u'(t)|^2\, dt,$$

for all $u \in W_2^1(I)$ such that $u(t_1) = u(t_2) = ... = u(t_n) = 0$.

Proof. Let $\varphi \in C_0^\infty(\mathbb{R})$ such that $\varphi(t) = 0$ if $|t| \ge 1$, $\varphi \ge 0$, and $\int_{\mathbb{R}} \varphi(t)\, dt = 1$. For $\varepsilon > 0$, let $\varphi_\varepsilon(t) = \frac{1}{\varepsilon}\varphi(\frac{t}{\varepsilon})$. Then $\varphi_\varepsilon(t) = 0$ if $|t| \ge \varepsilon$ and $\int_{\mathbb{R}} \varphi_\varepsilon(t)\, dt = 1$. Extend ν to $\mathbb{R}$ by $\nu(J) = 0$ for $J = \mathbb{R} \setminus \overline{I}$. Let $\nu_\varepsilon := \nu * \varphi_\varepsilon$, i.e.,

$$d\nu_\varepsilon(t) = \left(\int_{\mathbb{R}} \varphi_\varepsilon(t - y)\, d\nu(y)\right) dt.$$

Then $\operatorname{supp} \nu_\varepsilon \subseteq I_\varepsilon$, where $I_\varepsilon := [-\varepsilon, l + \varepsilon]$. By Lemma 3.4, for any $n \in \mathbb{N}$ there exists a partition $P_\varepsilon = \{t_0^\varepsilon, ..., t_n^\varepsilon\}$ of I_ε such that $|P_\varepsilon| = n$ and

$$\int_{I_\varepsilon} |u_\varepsilon(t)|^2 d\nu_\varepsilon(t) \le \frac{l}{n^2}\nu_\varepsilon(I_\varepsilon) \int_{I_\varepsilon} |u_\varepsilon'(t)|^2\, dt, \tag{3.5}$$

for all $u_\varepsilon \in W_2^1(I_\varepsilon)$ such that $u(t_1^\varepsilon) = ... = u(t_n^\varepsilon) = 0$.

Let

$$\xi(x) := \frac{l + 2\varepsilon}{l}x - \varepsilon.$$

Then

$$\xi^{-1}(y) = \frac{l}{l + 2\varepsilon}(y + \varepsilon)$$

and

$$\xi : I \longrightarrow I_\varepsilon, \quad \xi^{-1} : I_\varepsilon \longrightarrow I.$$

Let

$$t_k = \xi^{-1}\left(t_k^\varepsilon\right), \quad k = 0, ..., n.$$

Take any $u \in W_2^1(I)$ such that $u(t_1) = ... = u(t_n)$. Consider

$$u_\varepsilon(y) := u(\xi^{-1}(y)).$$

Then $u_\varepsilon \in W_2^1(I_\varepsilon)$ and $u_\varepsilon(t_1^\varepsilon) = ... = u_\varepsilon(t_n^\varepsilon) = 0$, so (3.5) holds.
Now,

$$\begin{aligned}
\nu_\varepsilon(I_\varepsilon) &= \int_{I_\varepsilon}\int_{\mathbb{R}} \varphi_\varepsilon(t-y)d\nu(y)\,dt = \int_{\mathbb{R}}\int_{I_\varepsilon} \varphi_\varepsilon(t-y)\,dtd\nu(y) \\
&= \int_{\overline{I}}\int_{I_\varepsilon} \varphi_\varepsilon(t-y)\,dtd\nu(y) = \int_{\overline{I}}\int_{\mathbb{R}} \varphi_\varepsilon(t-y)\,dtd\nu(y) \\
&= \int_{\overline{I}} d\nu(y) = \nu\left(\overline{I}\right), \qquad (3.6)
\end{aligned}$$

$$\begin{aligned}
\int_{I_\varepsilon} |u_\varepsilon'(t)|^2\,dt &= \int_{I_\varepsilon} \left|\frac{d}{dt}u(\xi^{-1}(t))\right|^2 dt = \frac{l}{l+2\varepsilon}\int_I |u'(x)|^2\,dx \\
&\le \int_I |u'(x)|^2\,dx, \qquad (3.7)
\end{aligned}$$

$$\begin{aligned}
&\left|\int_{\overline{I}} |u(y)|^2\,d\nu(y) - \int_{I_\varepsilon} |u_\varepsilon(t)|^2\,d\nu_\varepsilon(t)\right| \\
&= \left|\int_{\mathbb{R}} |u(y)|^2\,d\nu(y) - \int_{\mathbb{R}} |u_\varepsilon(t)|^2\,d\nu_\varepsilon(t)\right| \\
&= \left|\int_{\mathbb{R}} |u(y)|^2\,d\nu(y) - \int_{\mathbb{R}} |u_\varepsilon(t)|^2 \int_{\mathbb{R}} \varphi_\varepsilon(t-y)d\nu(y)\,dt\right| \\
&= \left|\int_{\mathbb{R}} |u(y)|^2\,d\nu(y) - \int_{\mathbb{R}}\int_{\mathbb{R}} |u_\varepsilon(\tau+y)|^2\varphi_\varepsilon(\tau)d\tau\,d\nu(y)\right| \\
&\le \int_{\mathbb{R}}\int_{\mathbb{R}} \left||u(y)|^2 - |u_\varepsilon(\tau+y)|^2\right| \varphi_\varepsilon(\tau)d\tau d\nu(y) \\
&\le \max_{\substack{y\in\overline{I} \\ |\tau|\le\varepsilon}} \left||u(y)|^2 - |u_\varepsilon(\tau+y)|^2\right| \nu\left(\overline{I}\right),
\end{aligned}$$

$$\begin{aligned}
&|u(y)|^2 - |u_\varepsilon(\tau+y)|^2 = |u(y)|^2 - \left|u\left(\frac{l}{l+2\varepsilon}(y+\tau+\varepsilon)\right)\right|^2 \\
&\le \left(|u(y)| - \left|u\left(\frac{l(y+\tau+\varepsilon)}{l+2\varepsilon}\right)\right|\right)\left(|u(y)| + \left|u\left(\frac{l(y+\tau+\varepsilon)}{l+2\varepsilon}\right)\right|\right) \\
&\le 2\sqrt{|I|}\left|y - \frac{l}{l+2\varepsilon}(y+\tau+\varepsilon)\right|^{\frac{1}{2}} \|u'\|_{L^2}^2 \\
&= 2\sqrt{l}\sqrt{\frac{1}{l+2\varepsilon}}\underbrace{|2\varepsilon y - l\tau - l\varepsilon|^{\frac{1}{2}}}_{\le\sqrt{4l\varepsilon}} \|u'\|_{L^2}^2 \\
&\le 4\sqrt{l}\sqrt{\frac{l}{l+2\varepsilon}}\sqrt{\varepsilon}\|u'\|_{L^2}^2 \le 4\sqrt{l}\sqrt{\varepsilon}\|u'\|_{L^2}^2.
\end{aligned}$$

Hence

$$\left|\int_{\overline{I}} |u(y)|^2\, d\nu(y) - \int_{I_\varepsilon} |u_\varepsilon(t)|^2\, d\nu_\varepsilon(t)\right| \le 4\sqrt{l}\sqrt{\varepsilon}\|u'\|^2_{L^2}\nu\left(\overline{I}\right).$$

This combined with (3.5),(3.6), and (3.7) implies

$$\begin{aligned}\int_{\overline{I}} |u(y)|^2\, d\nu(y) &\le \int_{I_\varepsilon} |u_\varepsilon(t)|^2\, d\nu_\varepsilon(t) + 4\sqrt{l}\sqrt{\varepsilon}\|u'\|^2_{L^2}\nu\left(\overline{I}\right)\\ &\le \frac{l}{n^2}\,\nu\left(\overline{I}\right)\int_I |u'(x)|^2\, dx + 4\sqrt{l}\sqrt{\varepsilon}\nu\left(\overline{I}\right)\int_I |u'(x)|^2\, dx\\ &= \left(\frac{l}{n^2} + 4\sqrt{l}\sqrt{\varepsilon}\right)\nu\left(\overline{I}\right)\int_I |u'(x)|^2\, dx.\end{aligned}$$

It is now left to take $\varepsilon > 0$ such that $\frac{l}{n^2} + 4\sqrt{l}\sqrt{\varepsilon} \le c\,\frac{l}{n^2}$, i.e.

$$\varepsilon \le \left(\frac{c-1}{4n^2}\right)^2 l.$$

□

Lemma 3.6. *For every $y \in \mathbb{R}_+$, there exists $c > 1$ such that*

$$\lceil cy\rceil - 1 \le y\,,$$

where $\lceil x\rceil$ is the smallest integer not less than x.

Proof. Case 1: Suppose $y \in \mathbb{R}_+\backslash\mathbb{Z}_+$. Then there exists $l \in \mathbb{Z}_+$ such that

$$l < y < l + 1\,.$$

Take $c > 1$ such that

$$l < cy < l + 1\,.$$

Then

$$\lceil cy\rceil - 1 = l + 1 - 1 = l < y\,.$$

Case 2: Suppose $y \in \mathbb{Z}_+$. Take $c > 1$ such that

$$cy < y + 1\,.$$

Then

$$\lceil cy\rceil - 1 = y + 1 - 1 = y\,.$$

□

We will need the following estimate. For any $0 \le a < b$ and $u \in W^1_2([a, b])$,

$$\frac{|u(x)|^2}{|x|} \le C(\kappa)\left(\int_a^b |u'(t)|^2\, dt + \kappa\int_a^b \frac{|u(t)|^2}{|t|^2}\, dt\right), \quad \forall x \in [a, b], \tag{3.8}$$

where

$$C(\kappa) = \frac{1}{2\kappa}\left(1 + \sqrt{1+4\kappa}\,\frac{b^{\sqrt{1+4\kappa}} + a^{\sqrt{1+4\kappa}}}{b^{\sqrt{1+4\kappa}} - a^{\sqrt{1+4\kappa}}}\right) \tag{3.9}$$

(see [9, Appendix A]). In the case $a = 0$, one should take $x > 0$ and assume that $u(0) = 0$, since otherwise the right-hand side of the above inequality is infinite.

4. Proof of Theorem 2.1

Let

$$X := W_2^1(\mathbb{R}), \quad X_0 := \{u \in X \ : \ u(0) = 0\},$$
$$X_1 := \left\{u \in W^1_{2,\mathrm{loc}}(\mathbb{R}) \ : \ u(0) = 0, \ \int_{\mathbb{R}} |\, u'(x)\,|^2\, dx < \infty\right\}.$$

Then, $\dim(X/X_0) = 1$ and $X_0 \subset X_1$. Let $\mathcal{E}_{X,\mu}$, $\mathcal{E}_{X_0,\mu}$, and $\mathcal{E}_{X_1,\mu}$ denote the forms

$$\int_{\mathbb{R}} |\, u'(x)\,|^2\, dx - \int_{\mathbb{R}} |\, u(x)\,|^2\, d\mu(x)$$

on the domains $X \cap L^2(\mathbb{R}, d\mu)$, $X_0 \cap L^2(\mathbb{R}, d\mu)$ and $X \cap L^2(\mathbb{R}, d\mu)$ respectively. Then

$$N_-(\mathcal{E}_{\mathbb{R},\mu}) = N_-(\mathcal{E}_{X,\mu}) \le N_-(\mathcal{E}_{X_0,\mu}) + 1 \le N_-(\mathcal{E}_{X_1,\mu}) + 1 \tag{4.1}$$

(see (2.1)). An estimate for the right hand of (4.1) is presented in [9] (see also [11]) for the case when μ is absolutely continuous with respect to the Lebesgue measure. We follow a similar argument. It follows from Hardy's inequality (see, e.g., [6, Theorem 327]) that

$$\int_{\mathbb{R}} |u'(x)|^2\, dx + \kappa \int_{\mathbb{R}} \frac{|u(x)|^2}{|x|^2}\, dx \le \int_{\mathbb{R}} |u'(x)|^2\, dx + 4\kappa \int_{\mathbb{R}} |u'(x)|^2\, dx$$
$$= (4\kappa + 1) \int_{\mathbb{R}} |u'(x)|^2\, dx, \quad \forall u \in X_1, \quad \forall \kappa \ge 0.$$

Hence

$$N_-(\mathcal{E}_{X_1,\mu}) \le N_-(\mathcal{E}_{\kappa,\mu}), \tag{4.2}$$

where

$$\mathcal{E}_{\kappa,\mu}[u] := \int_{\mathbb{R}} |u'(x)|^2\, dx + \kappa \int_{\mathbb{R}} \frac{|u(x)|^2}{|x|^2}\, dx - (4\kappa + 1) \int_{\mathbb{R}} |u(x)|^2\, d\mu(x),$$
$$\mathrm{Dom}\,(\mathcal{E}_{\kappa,\mu}) = X_1 \cap L^2(\mathbb{R}, d\mu).$$

It follows from (4.1) and (4.2) that

$$N_-(\mathcal{E}_{\mathbb{R},\mu}) \le N_-(\mathcal{E}_{\kappa,\mu}) + 1. \tag{4.3}$$

Let

$$\mathbf{I}_n := [2^{n-1}, 2^n], \ n > 0, \ \mathbf{I}_0 := [-1, 1], \quad \mathbf{I}_n := [-2^{|n|}, -2^{|n|-1}], \ n < 0.$$

The variational principle (see (3.1)) implies

$$N_-(\mathcal{E}_{\kappa,\mu}) \le \sum_{n \in \mathbb{Z}} N_-(\mathcal{E}_{\kappa,\mu,n}), \tag{4.4}$$

where

$$\mathcal{E}_{\kappa,\mu,n}[u] := \int_{\mathbf{I}_n} |u'(x)|^2\, dx + \kappa \int_{\mathbf{I}_n} \frac{|u(x)|^2}{|x|^2}\, dx - (4\kappa + 1) \int_{\mathbf{I}_n} |u(x)|^2\, d\mu(x),$$
$$\mathrm{Dom}\,(\mathcal{E}_{\kappa,\mu,n}) = W_2^1(\mathbf{I}_n) \cap L^2(\mathbf{I}_n, d\mu), \ n \in \mathbb{Z} \setminus \{0\},$$
$$\mathrm{Dom}\,(\mathcal{E}_{\kappa,\mu,0}) = \{u \in W_2^1(\mathbf{I}_0) : \ u(0) = 0\} \cap L^2(\mathbf{I}_0, d\mu).$$

Let $n > 0$. For any $c > 1$ and $N \in \mathbb{N}$, by Lemma 3.5 there exists a subspace $\mathcal{L}_N \in \mathrm{Dom}\,(\mathcal{E}_{\kappa,\mu,n})$ of co-dimension N such that

$$\int_{\mathbf{I}_n} |u(x)|^2\, d\mu(x) \le c\left(\frac{|\mathbf{I}_n|}{N^2}\,\mu\left(\mathbf{I}_n\right)\right)\int_{\mathbf{I}_n} |u'(x)|^2\, dx, \quad \forall u \in \mathcal{L}_N.$$

If

$$c(4\kappa + 1)\,\frac{|\mathbf{I}_n|}{N^2}\,\mu\left(\mathbf{I}_n\right) \le 1,$$

then $\mathcal{E}_{\kappa,\mu,n}[u] \ge 0$, $\forall u \in \mathcal{L}_N$, and $N_-(\mathcal{E}_{\kappa,\mu,n}) \le N$. Let

$$\mathcal{A}_n := \int_{\mathbf{I}_n} |x|\, d\mu(x),\ n \ne 0, \quad \mathcal{A}_0 := \int_{\mathbf{I}_0} d\mu(x).$$

Since $|\mathbf{I}_n| \int_{\mathbf{I}_n} d\mu(x) \le \mathcal{A}_n$, $n \ne 0$, it follows from the above that

$$c(4\kappa + 1)\mathcal{A}_n \le N^2 \implies N_-(\mathcal{E}_{\kappa,\mu,n}) \le N.$$

Hence

$$N_-(\mathcal{E}_{\kappa,\mu,n}) \le \left\lceil \sqrt{c(4\kappa + 1)\mathcal{A}_n} \right\rceil, \tag{4.5}$$

where $\lceil\cdot\rceil$ denotes the ceiling function, i.e. $\lceil a \rceil$ is the smallest integer not less than a. Suppose $\mathrm{supp}\mu \cap \mathbf{I}_n \ne \{2^{n-1}\}$, i.e., $\mu|_{\mathbf{I}_n} \ne \mathrm{const}\,\delta_{2^{n-1}}$. Then

$$|\mathbf{I}_n| \int_{\mathbf{I}_n} d\mu(x) < \mathcal{A}_n\ .$$

Take $c > 1$ such that

$$c|\mathbf{I}_n| \int_{\mathbf{I}_n} d\mu(x) \le \mathcal{A}_n\,.$$

Then applying Lemma 3.5 with this c implies

$$N_-(\mathcal{E}_{\kappa,\mu,n}) \le \left\lceil \sqrt{(4\kappa + 1)\mathcal{A}_n} \right\rceil. \tag{4.6}$$

If $\mu|_{\mathbf{I}_n} = \mathrm{const}\,\delta_{2^{n-1}} \ne 0$, then

$$\int_{\mathbf{I}_n} |u(x)|^2\, d\mu(x) = 0$$

on the subspace of co-dimension one consisting of functions $u \in W^1_2(\mathbf{I}_n)$ such that $u(2^{n-1}) = 0$, and clearly (4.6) holds. Finally, if $\mu|_{\mathbf{I}_n} = 0$, then (4.6) takes the form $0 \le 0$.

If $\mu|_{\mathbf{I}_n} \ne 0$, the right-hand side of (4.6) is at least 1, so one cannot feed it straight into (4.4). One needs to find conditions under which $N_-(\mathcal{E}_{\kappa,\mu,n}) = 0$. By (3.8), we have that

$$\begin{aligned}\int_{\mathbf{I}_n} |u(x)|^2\, d\mu(x) &\le C_0(\kappa)\int_{\mathbf{I}_n} |x|\, d\mu(x)\left(\int_{\mathbf{I}_n} |u'(x)|^2\, dx + \kappa\int_{\mathbf{I}_n} \frac{|u(x)|^2}{|x|^2}\, dx\right)\\ &= \mathcal{A}_n C_0(\kappa)\left(\int_{\mathbf{I}_n} |u'(x)|^2\, dx + \kappa\int_{\mathbf{I}_n} \frac{|u(x)|^2}{|x|^2}\, dx\right)\end{aligned}$$

for all $u \in W^1_2(\mathbf{I}_n)$, where

$$C_0(\kappa) = \frac{1}{2\kappa}\left(1 + \sqrt{1+4\kappa}\,\frac{2^{\sqrt{1+4\kappa}}+1}{2^{\sqrt{1+4\kappa}}-1}\right)$$

(cf. (3.9)).

Hence $N_-(\mathcal{E}_{\kappa,\mu,n}) = 0$, i.e. $\mathcal{E}_{\kappa,\mu,n}[u] \ge 0$, provided $\mathcal{A}_n \le \Phi(\kappa)$, where

$$\Phi(\kappa) := \frac{1}{(4\kappa+1)C_0(\kappa)} = \frac{2\kappa}{4\kappa+1}\left(1 + \sqrt{4\kappa+1}\,\frac{2^{\sqrt{4\kappa+1}}+1}{2^{\sqrt{4\kappa+1}}-1}\right)^{-1}.$$

The above estimates for $N_-(\mathcal{E}_{\kappa,\mu,n})$ clearly hold for $n < 0$ as well, but the case $n = 0$ requires some changes. Since $u(0) = 0$ for any $u \in \mathrm{Dom}\,(\mathcal{E}_{\kappa,\mu,0})$, one can use the same argument as the one leading to (4.5), but with two differences: a) $\mathcal{L}_N$ can be chosen to be of co-dimension $N-1$, and b) $|\mathbf{I}_0| \int_{\mathbf{I}_0} d\mu(x) = 2\mathcal{A}_0$. This gives the following analogue of (4.5)

$$N_-(\mathcal{E}_{\kappa,\mu,0}) \le \left\lceil \sqrt{2c(4\kappa+1)\mathcal{A}_0}\right\rceil - 1\,.$$

for any $c > 1$. We can choose $c > 1$ such that

$$N_-(\mathcal{E}_{\kappa,\mu,0}) \le \sqrt{2(4\kappa+1)\mathcal{A}_0}$$

(see Lemma 3.6). It is also easy to see that the implication $\mathcal{A}_n \le \Phi(\kappa) \implies N_-(\mathcal{E}_{\kappa,\mu,n}) = 0$ remains true for $n = 0$. Now it follows from (4.3) and (4.4) that

$$N_-(\mathcal{E}_{\mathbb{R},2\mu}) \le 1 + \sum_{\{n\in\mathbb{Z}\setminus\{0\}:\ \mathcal{A}_n > \Phi(\kappa)\}} \left\lceil \sqrt{(4\kappa+1)\mathcal{A}_n}\right\rceil + \sqrt{2(4\kappa+1)\mathcal{A}_0}\,, \tag{4.7}$$

and one can drop the last term if $\mathcal{A}_0 \le \Phi(\kappa)$. The presence of the parameter κ in this estimate allows a degree of flexibility. In order to decrease the number of terms in the sum in the right-hand side, one should choose κ in such a way that $\Phi(\kappa)$ is close to its maximum. A Mathematica calculation shows that the maximum is approximately 0.092 and is achieved at $\kappa \approx 1.559$. For values of κ close to 1.559, one has

$$\mathcal{A}_n > \Phi(\kappa) \implies \sqrt{(4\kappa+1)\mathcal{A}_n} > \sqrt{(4\kappa+1)\Phi(\kappa)} \approx 0.816.$$

Since $\lceil a\rceil \le 2a$ for $a \ge 1/2$, (4.7) implies

$$N_-(\mathcal{E}_{\mathbb{R},\mu}) \le 1 + 2\sqrt{(4\kappa+1)} \sum_{\mathcal{A}_n > \Phi(\kappa)} \sqrt{\mathcal{A}_n}$$

with $\kappa \approx 1.559$. Hence

$$N_-(\mathcal{E}_{\mathbb{R},\mu}) \le 1 + 5.38 \sum_{\{n\in\mathbb{Z},\ \mathcal{A}_n > 0.092\}} \sqrt{\mathcal{A}_n}\,.$$

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