

EHRENFUCHT-FRAÏSSÉ GAMES ON A CLASS OF SCATTERED LINEAR ORDERS

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ABSTRACT. Two structures A and B are n -equivalent if player II has a winning strategy in the n -move Ehrenfeucht-Fraïssé game on A and B . In earlier papers we studied n -equivalence classes of ordinals and coloured ordinals. In this paper we similarly treat a class of scattered order-types, focussing on monomials and sums of monomials in ω and its reverse ω^* .

1. Introduction

In [5] we studied the equivalence of finite coloured linear orders up to level n in an Ehrenfeucht-Fraïssé game, written as \equiv_n , which means that player II has a winning strategy in this game, as well as making some remarks about the infinite case. We gave some bounds for the minimal representatives in the finite case, and the infinite case for up to 2 moves. These results were extended in [6] to all coloured ordinals, in the monochromatic case giving a precise list of optimal representatives, and in the coloured case giving bounds, and in [7] some of the bounds for the finite case were improved for 3 moves, and additional details about the classification given for the 2-move case.

In this paper we tackle a class of linear orders which need not be well-ordered, where things are considerably more complicated. This is a subclass of the so-called ‘scattered’ linear orders, being those which do not embed the order-type of the rational numbers. According to Hausdorff’s characterization, these may be built up from 0 and 1 by means of sums over ordinals or reverse ordinals; see Theorem 2.2. A finer analysis of the class of scattered linear orderings in terms of finite sequences of finite ordinal-labelled trees, is given in [2] (concentrating on characterizing scattered orders up to mutual embeddability—‘equimorphism’). Even this class is too wide for us to analyze at this stage, and we restrict attention to ‘monomials’, which are products of (possibly infinitely many) copies of ω or its reverse ω^* , and certain sums of these. First we briefly recall the required definitions.

For a linear order $(A, <)$ we just write A provided that the ordering is clear. In the n -move Ehrenfeucht-Fraïssé game $G_n(A, B)$ on linear orders A and B (or indeed any relational structures) players I and II play alternately, I moving first. On each move I picks an element of either structure (his choice does not have to be from the same structure on every move), and II responds by choosing an element of the other structure. After n moves, I and II between them have chosen elements a_1, a_2, \dots, a_n of A , and b_1, b_2, \dots, b_n of B , and player II *wins* if the map taking a_i to b_i for each i is an isomorphism of induced substructures (that is, it preserves the ordering), and player I wins otherwise. It is not required that the a_i are all distinct, or that the b_i are all distinct (though if $a_i = a_j$ but $b_i \neq b_j$ then player II will automatically lose). There is no advantage to player I in repeating a move he has played earlier, but we do need to consider this option in view of ‘2-phase’

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games which arise later, where moves which are distinct on points may coincide on ‘blocks’. As we wish to consider varying values of n , we may also write $G(A, B)$ for the family of games $\{G_n(A, B) : n \geq 1\}$.

Intuitively, I is trying to demonstrate that there is some difference between the structures, while player II is trying to show that they are at least reasonably similar. We say that A and B are n -equivalent and write $A \equiv_n B$, if II has a winning strategy. It is easy to see that \equiv_n is an equivalence relation, and it is standard that for any n , there are only finitely many n -equivalence classes, so it is natural to enquire what their optimal representatives may be. The problem for general orderings seems to be quite hard, but with special conditions on the type of ordering, or the number of moves, some results can be obtained. For ordinals, the notion of ‘optimality’ makes sense since we may just choose the least representative, but in the orders we examine in this paper, this is not clear.

It is easy to see that structures A and B are elementarily equivalent, written $A \equiv B$, if and only if $A \equiv_n B$ for all n , so we may regard n -equivalence as a natural approximation to elementary equivalence. We remark that the downward Löwenheim-Skolem Theorem reduces the problem of the classification of linear orderings up to elementary equivalence to the same problem for countable linear orderings, so if we are trying to characterize optimal representatives for n -equivalence, we may also restrict to the countable case.

If A and B are linear orders, then $A + B$ stands for the concatenation of A and B , that is, we first assume (by replacing by copies if necessary) that A and B are disjoint, and we place all members of A to the left of all members of B . As a generalization of this, we may write $\sum\{A_i : i \in I\}$ for the concatenation of a family of linear orders $\{A_i : i \in I\}$ indexed by a linear ordering I . We write $A \cdot B$ for the anti-lexicographic product, B ‘copies of’ A , to accord with the customary use for ordinals (and unlike [6], where *lexicographic* products are used). A linear ordering is said to be *scattered* if the order-type of the rational numbers does not embed in it. In [8] Corollary 2.1.8 it is shown that if A and B are scattered, then so are $A + B$ and $A \cdot B$. Following [8], we denote by ζ the order-type of the integers \mathbb{Z} , which can also be construed as $\omega^* + \omega$, and by η the order-type of the rational numbers \mathbb{Q} .

The following straightforward result will be used without explicit reference.

Lemma 1.1. (i) If $A \equiv_n B$, then $X + A + Y \equiv_n X + B + Y$ and $X \cdot A \cdot Y \equiv_n X \cdot B \cdot Y$.
(ii) If $A_i \equiv_n B_i$ for each $i \in I$, then $\sum\{A_i : i \in I\} \equiv_n \sum\{B_i : i \in I\}$.

Every ordinal α can be written in the form $\alpha = \omega^\omega \cdot \alpha_1 + \alpha_2$ where $\alpha_2 < \omega^\omega$ and in terms of this representation, the following theorem of Mostowski and Tarski [3] helps in understanding ordinals up to elementary equivalence.

Lemma 1.2. Let $\alpha = \omega^\omega \cdot \alpha_1 + \alpha_2$ and $\beta = \omega^\omega \cdot \beta_1 + \beta_2$ be ordinals, where $\alpha_2, \beta_2 < \omega^\omega$. Then α is elementarily equivalent to β if and only if $\alpha_2 = \beta_2$ and either $\alpha_1 = \beta_1 = 0$ or $\alpha_1, \beta_1 > 0$.

As remarked in [8], this result enables us to conclude that ‘the set of all ordinals less than $\omega^\omega \cdot 2$ forms a *complete and irredundant* set of representatives of the elementary equivalence classes of well-orderings’. The following more precise result of Mostowski and Tarski is also needed (see [8] for a proof):

Lemma 1.3. For any $n > 0$, and ordinal $\beta > 1$, (i) $\omega^n \equiv_{2n} \omega^n \cdot \beta$, (ii) $\omega^n \not\equiv_{2n+1} \omega^n \cdot \beta$.

From Lemma 1.2 we know that if α and β are distinct ordinals which are both $< \omega^\omega$, then $\alpha \not\equiv \beta$. Later in the paper we require the following more precise

information about the level at which this elementary inequivalence is shown. Note that the estimate of the number of moves required is given in terms of the Cantor normal form for α , even though β may have a longer Cantor normal form (but this does not matter).

Lemma 1.4. *If $\alpha = \omega^{m_0} + \omega^{m_1} + \dots + \omega^{m_{k-1}}$ where $m_0 \geq m_1 \geq \dots \geq m_{k-1}$, and $\beta < \alpha$, then $\alpha \not\equiv_{2m_0+k} \beta$.*

Proof. We use induction on k . If $k = 1$ then $\alpha = \omega^{m_0}$. If $\beta = 0$, player I wins in one move by playing any point of α . Otherwise, there is $b_1 \in \beta$ such that $\beta^{\geq b_1} \cong \omega^l$ for some l (possibly 0), which player I plays on his first move. Since $\beta < \alpha$, $l < m_0$. Let $a_1 \in \alpha$ be II's reply. Then $\alpha^{\geq a_1} \cong \omega^{m_0}$ and $\beta^{\geq b_1} \cong \omega^l$, so player I can win in at most $2l + 1 \leq 2m_0 - 1$ more moves by Lemma 1.3(ii), if $l > 0$ (if $l = 0$, then b_1 is greatest in β , so I wins in 1 more move by playing any point $a_2 > a_1$).

For the induction step, on his first move, player I plays the first point a_1 of ω^{m_1} , and we let b_1 be II's reply. If b_1 is the least point of β , then I wins on the next move by playing the least point of α . Otherwise, let $b < b_1$ be such that for some l , $[b, b_1) \cong \omega^l$. Since $\beta < \alpha$, $l \leq m_0$. If $l < m_0$, player I plays $b_2 = b$, and whatever $a_2 < a_1$ II plays, $[a_2, a_1) \cong \omega^{m_0}$ and $[b_2, b_1) \cong \omega^l$, so I wins in $2l + 1 \leq 2m_0 - 1$ more moves by Lemma 1.3(ii), or by the above argument if $l = 0$. Otherwise, $l = m_0$ and hence if $\alpha' = \omega^{m_1} + \dots + \omega^{m_{k-1}} \cong \alpha^{\geq a_1}$ and $\beta' = \beta^{\geq b_1}$, $\beta' < \alpha'$. By the induction hypothesis, player I can now win on the right in at most $2m_1 + k - 1$ more moves, and since $m_1 \leq m_0$, this makes at most $2m_0 + k$ in all. ■

The next lemma tells us how reversal works with respect to sums and products, where for any A , A^* stands for the linear ordering obtained by reversing the ordering on A .

Lemma 1.5. *If A , B , and I are linear orderings, then*

- (i) $(A + B)^* = B^* + A^*$,
- (ii) $(A \cdot B)^* = A^* \cdot B^*$,
- (iii) $(\sum\{A_i : i \in I\})^* = \sum\{(A_i)^* : i \in I^*\}$ where A_i are linear orderings.

In [6], a classification is given of all ordinals up to n -equivalence, which is based on their Cantor normal form representation. In this paper we give some results aimed at classifying all scattered linear orderings up to elementary equivalence and n -equivalence. Since this forms a large and complicated class, we restrict attention to ones which can be reasonably easily constructed. We shall see (in Theorem 2.3, Corollary 3.7 and Theorem 4.5, and using the downward Löweinheim-Skolem theorem) that there are exactly 2^{\aleph_0} elementary equivalence classes of scattered orders. We principally focus on ‘monomials’, and sums of these indexed by a finite set, or ω , ω^* , or ζ .

Definition 1.6. *A **monomial** is a linear ordering which is the restricted anti-lexicographic product of a non-empty well-ordered sequence of orderings each of which is ω , ω^* , ζ , or a non-zero natural number. It is said to be a **countable monomial** if it is the product of a countable such sequence.*

Notice that we require the family to be well-ordered, and this is so that when we order it anti-lexicographically, it is scattered, see Lemma 2.4 below. (We only need consider anti-lexicographic products since lexicographic ones are isomorphic to the anti-lexicographic products obtained by reversing the order of the terms; the order the terms are taken in, and whether the product is lexicographic or anti-lexicographic, crucially affects its behaviour.) In addition, in order to obtain a linear order at all, the product needs to be taken to be ‘restricted’, which means that it consists of the members of the product having ‘finite support’. The general

notion is this: let I and X_i for all $i \in I$ be linearly ordered sets and assume that $0 \in X_i$ for all $i \in I$ (where 0 may actually lie in X_i , or is some arbitrarily chosen ‘default value’). The infinite restricted product $\prod_{i \in I} X_i$ then stands for the set of functions $f : I \rightarrow \bigcup_{i \in I} X_i$ such that $f(i) \in X_i$ for all i , and $\{i : f(i) \neq 0\}$ is finite. We say that f has *finite support*. The order is taken to be *anti-lexicographic*, that is, by last difference (which extends the definition for the product of two linear orders to the general case). This means that for $f, g \in \prod_{i \in I} X_i$, $f < g$ if $(\exists i)(f(i) < g(i) \wedge ((\forall j > i)f(j) = g(j)))$. The hypothesis of finiteness of the support guarantees that such i always exists. For example, $\omega \cdot \omega \cdot \omega \cdots$ is a scattered linear ordering (just equal to the usual ordinal power ω^ω) when ordered anti-lexicographically but $\cdots \omega \cdot \omega \cdot \omega$ is dense as is seen in Lemma 2.4.

The paper is organized as follows. In section 2 we give further preliminary results needed, leading up to the ‘Two-phase Lemma’ 2.9. In section 3 we give our main results concerning n -equivalence of monomials. Theorems 3.4 and 3.5 characterize the optimal length of equivalence of monomials which are finite products of ω and ω^* . This is extended to monomials involving infinite terms, or infinitely many terms, or both, or involving a finite term or ζ . Finally in section 4 we consider sums of finitely many, ω , ω^* , or ζ monomials. Because of the complications involved, we consider arbitrary sums just for powers of ω or ω^* , and sums of just *two* more general monomials. These results should be enough to illustrate the issues arising.

2. Preliminary results

Scattered linear orderings may be analyzed in terms of *Hausdorff rank* as given in [8] (where it was called *F-rank*). Here we just treat the countable case.

Definition 2.1. Let $V = \bigcup\{V_\alpha : \alpha < \omega_1\}$ where

- (i) $V_0 = \{0, 1\}$,
- (ii) if $\alpha > 0$, V_α is the set of all linear orderings which may be written in the form $\sum\{L_i : i \in I\}$ where $I \cong \omega, \omega^*, \zeta = \omega^* + \omega$, or finite n , and each L_i lies in $\bigcup\{V_\beta : \beta < \alpha\}$,
- (iii) the Hausdorff rank of L is the least α , if any, such that $L \in V_\alpha$.

The following theorem characterizes all countable scattered linear orderings.

Theorem 2.2 (Hausdorff,[8]). *A countable linear ordering L is in V if and only if it is a countable scattered linear ordering.*

Thus Lemma 1.2 tells us that every ordinal is elementarily equivalent to an ordinal with Hausdorff rank at most $\omega + 1$.

In terms of Hausdorff rank, we may view ω and ω^* as the simplest (infinite) scattered orderings, and ones built up from these and finite orders by ω - (or ω^* -) sums as the next most complicated. Even here, we have 2^{\aleph_0} elementarily inequivalent examples, as is shown in the following well-known result. We give two other examples of such families later in the paper, in Corollary 3.7 for (infinite) monomials, and in Theorem 4.5 for ω -sums of powers of ω or ω^* .

Theorem 2.3. *There are 2^{\aleph_0} distinct equivalence classes of countable scattered linear orderings modulo \equiv .*

Proof. For each $X \subseteq \mathbb{N}$ we shall find $L(X)$ so that if $X \neq Y$, then $L(X) \not\equiv L(Y)$. We let $M_n(X)$ be of order-type $\omega^* + \omega + n + 2$ if $n \in X$ and of order-type $\omega^* + \omega$ if $n \notin X$; and let $L(X) = \sum\{M_n(X) : n < \omega\}$, which is obtained by concatenating $M_0(X), M_1(X), M_2(X), \dots$ in that order. Let $X, Y \subseteq \mathbb{N}$ be such that $X \neq Y$. Let $n \in X \setminus Y$ (if $X \setminus Y \neq \emptyset$, or $n \in Y \setminus X$ otherwise). Then player I has a winning strategy in the $(n+4)$ -move game on $L(X)$ and $L(Y)$. For player I chooses in order

the final $n+2$ elements of $M_n(X)$ for his first $n+2$ turns. He then considers player II's $n+2$ responses b_1, b_2, \dots, b_{n+2} in $L(Y)$. If, for some i , there is an element of $L(Y)$ between b_i and b_{i+1} then player I chooses that element on his $(n+3)$ rd turn and player II loses immediately. Otherwise, since there are no maximal discrete sequences of exactly $n+2$ elements in $L(Y)$, either b_{n+2} has an immediate successor, or b_1 has an immediate predecessor in $L(Y)$. Player I chooses such an element b_{n+3} ; suppose without loss of generality, that $b_{n+2} < b_{n+3}$. Then player II must choose an element a_{n+3} of $L(X)$ beyond the last point a_{n+2} of $M_n(X)$. But then player I chooses an element between a_{n+2} and a_{n+3} , and as there is no element between b_{n+2} and b_{n+3} , player II loses. This therefore provides a winning strategy for I. Thus if X and Y are distinct subsets of \mathbb{N} , then $L(X) \not\equiv L(Y)$, so there are 2^{\aleph_0} distinct equivalence classes of countable scattered linear orderings modulo \equiv . ■

The next result shows why we insist that products are taken over well-ordered families (under the anti-lexicographic order).

Lemma 2.4. *For any non-well-ordered set I and linear orders $\alpha_i > 1$ for all $i \in I$, $\prod_{i \in I} \alpha_i$ is not scattered.*

Proof. Let $\beta = \prod_{i \in I} \alpha_i$. We look at β as a linear ordering of functions as defined above. Since I is not well-ordered, it has an infinite subset I' ordered in type ω^* and we show that the set X of elements of β whose support is contained in I' is densely ordered. Let $f, g \in X$ be such that $f < g$. Let $t \in I$ be such that for $i > t$, $f(i) = g(i)$ and $f(t) < g(t)$. Since $f(t) \neq g(t)$, $t \in I'$ so there is $s < t$ in I' outside the union of the supports of f and g .

In the first case there is $a > 0$ in α_s , in which case we let

$$h(i) = \begin{cases} f(i) & \text{if } i \neq s \\ a & \text{if } i = s \end{cases}$$

Then the greatest (only) point at which f and h differ is s , and since $f(s) = 0 < a = h(s)$, we have $f < h$. Since t is the greatest point at which h and g differ and $h(t) = f(t) < g(t)$, also $h < g$. If 0 is the greatest in α_s (for example if $\alpha_s = \omega^*$) then pick $a < 0$ in α_s and instead let

$$h(i) = \begin{cases} g(i) & \text{if } i \neq s \\ a & \text{if } i = s \end{cases}$$

and once more, $f < h < g$. ■

Although ζ is allowed as a term in the definition of ‘monomial’, we now show that its role is insignificant.

Lemma 2.5. (i) *For any $B \neq \emptyset$, $\zeta \cdot B \equiv \zeta$ and $\omega \equiv \omega + \zeta \cdot B$.*

(ii) *For any orderings A and B for which A has a last element, $\omega \cdot A + \zeta \cdot B \equiv \omega \cdot A$.*

Proof. (i) Take any $n \geq 1$ and we show that $\zeta \cdot B \equiv_n \zeta$. Since ζ is infinite ‘in both directions’, player II may play in such a way that if $a_1 < a_2 < \dots < a_r$ and $b_1 < b_2 < \dots < b_r$ are the first r moves by the two players in $\zeta \cdot B$ and ζ , so that for each j , a_j and b_j are the moves played on the same move (not necessarily the j th), then for $1 \leq j < r$, $|(a_j, a_{j+1})|$ and $|(b_j, b_{j+1})|$ are either equal, or are both $\geq 2^{n-r} - 1$. (Of course, $(-\infty, a_1)$, $(-\infty, b_1)$, (a_r, ∞) , (b_r, ∞) are all infinite.) It is easy for player II to play so that this statement is true (since the required minimum length of each interval is essentially halved at each stage), and when $r = n - 1$, there is still a point available for him to play.

For ω and $\omega + \zeta \cdot B$, in n moves, II follows a similar strategy, in addition ensuring that $|(-\infty, a_1)|$ and $|(-\infty, b_1)|$ are either equal, or both $\geq 2^{n-r} - 1$.

(ii) Write $X = \omega \cdot A + \zeta \cdot B$, $Y = \omega \cdot A$, and $A = C + 1$. Then $X = \omega(C + 1) + \zeta \cdot B$ and $Y = \omega(C + 1)$. Thus $X = \omega \cdot C + \omega + \zeta \cdot B$ and $Y = \omega \cdot C + \omega$, and the result follows from (i) (and Lemma 1.1). ■

We note that part (i) applies even if B is not scattered, though in this paper we restrict to the scattered case. We also note that (ii) may be false without the assumption that A has a last element. For instance, $\omega^2 + \zeta \not\equiv_4 \omega^2$. For on his first move, I plays a_1 in ζ . Let $b_1 \in \omega^2$ be II's reply. Now I plays the first point b_2 of the copy of ω greater than b_1 , and II's reply a_2 must be in ζ . Now I plays the predecessor a_3 of a_2 in ζ ; let $b_3 \in [b_1, b_2)$ be II's reply. Since b_2 is a limit ordinal, I may now play in (b_3, b_2) , and II cannot respond.

In any monomial, we group together like terms, so that we can alternatively define a monomial to be a (restricted, anti-lexicographic) well-ordered product of orders of the form ω^α , $(\omega^*)^\beta$, ζ^γ , m^δ for $m \in \omega$, $m \geq 2$, and non-zero ordinals α , β , γ , δ , no two consecutive ones of the same type, and these are then referred to as the *terms* of the monomial. Any monomial of interest to us will fall into one of three categories, namely a *finite monomial product*, an ω -*monomial product* or a *transfinite monomial product* (where the last one is indexed by an ordinal greater than ω). We refer to all finite ordinals as *trivial* monomials. For example, $\omega^2 \cdot \omega^{*2} \cdot \omega^3 \cdot \zeta \cdot 4$ is a finite monomial product and $(\omega^*)^2 \cdot \omega^5 \cdot \zeta \cdot (\omega^*)^2 \cdot \omega^5 \cdot \zeta \cdots \omega^2 \cdot \zeta^3 \cdots$ is a transfinite monomial product. We refer to α , β and γ as the *exponents* of the terms ω^α , ω^β , and ζ^γ respectively. A monomial in which ζ appears is called a ζ -*product*.

Corollary 2.6. *Any ζ -product is elementarily equivalent to one of the form $A \cdot \zeta$, where ζ does not occur in A .*

Proof. Let A be a monomial in which ζ occurs. By considering the first occurrence of ζ , we may write $A = A_1 \cdot \zeta \cdot A_2$, where ζ does not occur in A_1 . By Lemma 2.5, $\zeta \cdot A_2 \equiv \zeta$, so $A \equiv A_1 \cdot \zeta$. ■

Now there are various standard elementary equivalences which are well known or easily established, such as $\zeta \equiv \zeta \cdot B$ given in Lemma 2.5. Since we are focussing on optimal representatives for countable scattered linear orders, these standard equivalences lead us to rule out certain monomials from consideration (and a full treatment would reduce a general monomial to one of the simplified ones in our list). We do not give a completely precise definition of *simple*, or *optimal* form. The idea is that it should at least be a canonical representative of an equivalence class under \equiv (or some \equiv_n), and should be of least possible Hausdorff rank. If it exists, then it should be possible to recognize it as such, but it might be hard to show that it does *not* exist. The principal restrictions which seem to be required are now given. We let Σ be the family of monomials A satisfying the following:

- (a) No finite ordinal appears as a term in A except on the extreme right.
(This is because, if $n \neq 0$ is finite and A is any non-trivial monomial, then $n \cdot A \equiv A$, since $n \cdot \omega = \omega$, $n \cdot \omega^* = \omega^*$ and $n \cdot \zeta = \zeta$.)
- (b) If ω^α or $(\omega^*)^\alpha$ is a term of A , then $\alpha \leq \omega$ (by Lemma 1.2).
- (c) Any ζ -product has the form $A \cdot \zeta$ where ζ does not appear in A .

The above discussion shows that our first main task is to classify the restricted monomials that involve only products of $(\omega^*)^\alpha$, ω^β and n where n appears as the last term of the monomial (we exclude ζ since it can increase rank by at most 1).

Finally in this section we give some basic lemmas about n -equivalence. The following result gives finer detail about the level of elementary equivalence, from which

Lemma 1.2 can be deduced, and is a slightly stronger version of the corresponding result of Mostowski and Tarski quoted in [6].

Lemma 2.7. *For any integer $n > 0$ and non-empty linear orders X and Y , such that either X and Y both have a least element, or neither do,*

- (i) $\omega^n \cdot X \equiv_{2n} \omega^n \cdot Y$,
- (ii) $\omega^n \not\equiv_{2n+1} \omega^n \cdot X$ if $|X| > 1$.

Proof. Note that we are just assuming that X and Y are linearly ordered, not necessarily well-ordered, so we cannot deduce the result from the one in [6] about ordinals.

We prove (i) by induction. If $n = 1$, player II may play so that I plays the least point of one of the orders if and only if II plays the least point of the other order, and then II can respond to any second move by I. (This shows why the hypotheses on X and Y are required; if X has a least element and Y does not, player I can win immediately by playing the first point of $\omega \cdot X$.) For the induction step, assuming the result for n , let $A = \omega^{n+1} \cdot X$, $B = \omega^{n+1} \cdot Y$, $A' = \omega^n \cdot X$, $B' = \omega^n \cdot Y$, so that $A = \omega \cdot A'$ and $B = \omega \cdot B'$. By induction hypothesis, $A' \equiv_{2n} B'$, so II has a winning strategy σ here. In the $2(n+1)$ -move game on A and B , he follows σ ‘on the copies of ω ’, making sure that on each move he plays the member of the current copy of ω corresponding to what I played.

Consider the first point at which II is unable to play the member of ω corresponding to what I has played. This can only happen in the last two moves. Without loss of generality, suppose that I’s move a lies in A , and let $a^- < a < a^+$ where a^- , a^+ are the closest points of A to a already played on left and right respectively ($a^- = -\infty$ or $a^+ = \infty$ are allowed if there are none such), and let b^- , b^+ be the corresponding points of B . Since II cannot respond, b^- and b^+ must lie in consecutive copies of ω (if there was another copy of ω in between, he could choose a suitable member of it; if b^- and b^+ are in the same copy, then so are a^- and a^+ , hence also a , and now II can certainly play the point corresponding to a). Hence $(b^-, b^+) \cong \omega + k$ for some finite k , and $(a^-, a^+) \cong \omega + \omega \cdot Z + k$ for some linear order Z (since the argument just given shows that a^- and a^+ lie in distinct copies). By the basis case, $\omega \equiv_2 \omega + \omega \cdot Z$, so also $\omega + k \equiv_2 \omega + \omega \cdot Z + k$, and II can play successfully for the last 2 moves.

(ii) is also proved by induction. On his first move, I plays the least member b_1 of a copy of ω^n (but not the first) in $\omega^n \cdot X$. Let $a_1 \in \omega^n$ be II’s response. If $a_1 = 0$, then I wins at once by playing to the left of his first move in $\omega^n \times X$. Otherwise he plays $a_2 < a_1$ in ω^n so that $[a_2, a_1] \cong \omega^k$ for some k , which must be $< n$ (it is possible that $k = 0$, as must happen for instance in the basis case, $n = 1$). Let II’s second move in $\omega^n \times X$ be b_2 . Then $(b_2, b_1) \cong \omega^n \times Z$ for some non-empty Z . If $(a_2, a_1) = \emptyset$ (that is, $k = 0$) then I wins on the third move by playing in between b_2 and b_1 in B . Otherwise $k > 0$, and $(a_2, a_1) \cong \omega^k$, and by induction hypothesis, I can win in the remaining $2n - 1$ moves by playing between a_2 and a_1 in A , and between b_2 and b_1 in B , since $2k + 1 \leq 2n - 1$. ■

Definition 2.8. *Let A and B be elementarily inequivalent linear orderings and n be a non-zero natural number. Then n is the optimal length of elementary equivalence of A and B if $A \equiv_n B$ and $A \not\equiv_{n+1} B$.*

In this case, we shall refer to a strategy that enables player I to win in $n + 1$ moves as an *optimal strategy* for player I, and a strategy that enables player II to win in n moves as an *optimal strategy* for player II. We denote the optimal length of elementary equivalence of a game on A and B by $l(A, B)$. We remark that $l(A, B)$ is a unique non-zero natural number unless A and B are elementarily equivalent (when we could write $l(A, B) = \infty$).

Lemma 2.9 (2-phase lemma). *Let A', B' be non-empty linear orders, m a positive integer, and $A = \omega^m \cdot A', B = \omega^m \cdot B'$, such that $l(A', B') = n$.*

(i) *If A', B' either both have a least element, or neither does, then $l(A, B) = 2m + n$.*

(ii) *If one of A', B' has a least element but the other does not, then $l(A, B) = 1$.*

Proof. Let σ and τ be optimal strategies for players I and II in the $(n + 1)$ -move and n -move games respectively on A' and B' .

We first give a winning strategy for player I in the $(2m + n + 1)$ -move game on A and B . For the first $n + 1$ moves, I plays according to σ on the ‘copies’ of ω^m in A , on each move playing the first point of the copy. Let’s assume that player II plays so that he hasn’t yet lost after these $n + 1$ moves. This means that the partial map determined by these $n + 1$ moves of the two players in A and B is order-preserving. Since σ is a winning strategy, the map cannot be (strictly) order-preserving on the copies, and this means that player II must have played a point of a copy of ω^m which contains a point that has already been played. Therefore, player II must at some point in these first $n + 1$ moves have played a point which is not the first in its copy of ω^m . Without loss of generality, suppose that the first such point b played by II lies in B , and this is in response to $a \in A$ played by I. Then for some $b' < b$, and $k < m$, $[b', b) \cong \omega^k$. Player I now plays b' as his next move. Let $a' \in A$ be II’s reply. Then $a' < a$ and the interval (a', a) has order-type of the form $\omega^m \cdot Z$ for some non-empty Z . If there is no point between b' and b (so that $k = 0$), I wins on the next move by playing in A between a' and a . Otherwise, $(b', b) \cong \omega^k$ and $(a', a) \cong \omega^m \cdot Z$, so by Lemma 2.7(ii), player I can win in the next $2k + 1$ moves, and $n + 2 + 2k + 1 \leq n + 2 + 2m - 1 = 2m + n + 1$.

Next we note that if A' has a least element and B' does not then the same applies to A and B , and I can win the game on A' and B' , and also the game on A and B , in 2 moves, by playing the least element of A' , or A as the case may be, on his first move; since B', B have no least, whatever II plays on his first move, I wins by playing to its left. Thus actually $l(A', B') = l(A, B) = 1$, giving (ii). Similarly if B' has a least but A' does not.

So now assume that A', B' either both have a least element, or neither does. Observe that we may assume that under τ , I plays the first element of one of A' and B' (if it exists) if and only if II plays the first element of the other structure. Indeed, on all moves except the last, II *must* play thus, since otherwise he loses on the next move as I will play to the left of the non-first element. And on his last move, there can be no disadvantage for II in doing so.

We can now give a winning strategy for player II in the $(2m + n)$ -move game on A and B . For the first n moves, player II uses τ on A' and B' , playing the corresponding point within each copy to that played by I. There are now $2m$ moves remaining. It suffices to show that II can win the game restricted to (a_1, a_2) in A and (b_1, b_2) in B , where a_1 and a_2 are consecutive points already played in A , and b_1 and b_2 are the corresponding points in B (or one or other may be $\pm\infty$ if the other is the greatest or least so far played), in the remaining moves. In other words, we have to see that $(a_1, a_2) \equiv_{2m} (b_1, b_2)$. Note that for some ordinals $\alpha_1, \alpha_2 < \omega^m$, and $a'_1, a'_2 \in A', b'_1, b'_2 \in B', a_1 = (\alpha_1, a'_1), a_2 = (\alpha_2, a'_2), b_1 = (\alpha_1, b'_1),$ and $b_2 = (\alpha_2, b'_2)$ (where these are now *ordered pairs*, and not intervals). Since τ is winning for II ‘on the blocks’ in n moves, $a'_1 = a'_2 \Leftrightarrow b'_1 = b'_2$, and in this case, the intervals (a_1, a_2) and (b_1, b_2) are actually equal (to the same ordinal $\leq \omega^m$), and therefore $(a_1, a_2) \equiv_{2m} (b_1, b_2)$. Otherwise, $a'_1 < a'_2$ and $b'_1 < b'_2$, and (a_1, a_2) and (b_1, b_2) are isomorphic to $\omega^m \times X + \alpha_2, \omega^m \times Y + \alpha_2$, for some non-empty X and Y , both having a least element, and by Lemma 2.7(i), these are $2m$ -equivalent. This analysis also applies to the case $a_1 = b_1 = -\infty$. Here by assumption on τ , a'_2 is least in $A' \Leftrightarrow b'_2$

is least in B' , in which case $(a_1, a_2) \cong \alpha_2 \cong (b_1, b_2)$. Since ω^m has no greatest, the corresponding problem does not arise on the right. ■

3. The class of monomials of countable scattered linear orderings

In this section we look at monomials of the forms

$$\begin{aligned} &M_0^{n_0} \cdot M_1^{n_1} \cdots M_{s-1}^{n_{s-1}}, \\ &M_0^{n_0} \cdot M_1^{n_1} \cdot M_2^{n_2} \cdots, \\ &\text{and } M_0^{n_0} \cdot M_1^{n_1} \cdots M_\omega^{n_\omega} \cdot M_{\omega+1}^{n_{\omega+1}} \cdots M_{\omega+t-1}^{n_{\omega+t-1}} \end{aligned}$$

(finite monomial, ω -monomial, transfinite monomial respectively) where each M_i is ω or ω^* and the two types alternate, and the n_i are non-zero. Mostly we suppose that the two monomials we are comparing begin with the same one of ω , ω^* , as otherwise things are rather easy, as explained by the first result.

Theorem 3.1. *Let $A = M_0^{n_0} \cdot M_1^{n_1} \cdot M_2^{n_2} \cdots M_{s-1}^{n_{s-1}}$ and $B = N_0^{n_0} \cdot N_1^{n_1} \cdots N_{t-1}^{n_{t-1}}$ where s, t, m_i, n_j are positive integers, and M_i alternate between ω and ω^* (or the other way round), and so do the N_j , and $M_0 = \omega \Leftrightarrow N_0 = \omega^*$. Then if $s = 1$ or $t = 1$ (or both), $l(A, B) = 1$, and otherwise, $l(A, B) = 2$.*

Proof. For the first part, let us suppose that $s = 1$ and $M_0 = \omega$, the other cases being similar. Then A has a least member and B does not. Here I plays the least member a_1 of A on his first move, and then to the left of whatever $b_1 \in B$ II plays, and wins on the second move.

If $s, t \geq 2$, then neither A nor B has a least or greatest element, and this ensures that II can win in 2 moves. To see that I can win in 3 moves, again assume for ease that $M_0 = \omega$, and I plays on his first move the least member a_1 of a copy of ω^{m_0} . II's reply b_1 lies in a copy of $(\omega^*)^{n_0}$, so has an immediate predecessor b_2 , which I plays on his second move. Since a_1 has no immediate predecessor in A , I wins on the third move by playing between II's moves a_2 and a_1 . ■

The next result is of use when we consider 'extra' parts of a monomial beyond the significant sections.

Theorem 3.2. *Let $A = M_0^{n_0} \cdot M_1^{n_1} \cdot M_2^{n_2} \cdots M_{s-1}^{n_{s-1}} \cdot X$ and $B = M_0^{n_0} \cdot M_1^{n_1} \cdots M_{s-1}^{n_{s-1}} \cdot Y$ where for all i , n_i is a non-zero finite number, X and Y are non-empty linear orders, and M_0, M_1, \dots alternate between ω and ω^* (or the other way round). Then*

(i) *if X and Y either both have least elements, or neither do, and similarly for greatest elements, then $A \equiv_{2(n_0+n_1+\dots+n_{s-1})} B$,*

(ii) *if $|X| > 1$ then*

$$M_0^{n_0} \cdot M_1^{n_1} \cdot M_2^{n_2} \cdots M_{s-1}^{n_{s-1}} \not\equiv_{2(n_0+n_1+\dots+n_{s-1})+1} M_0^{n_0} \cdot M_1^{n_1} \cdots M_{s-1}^{n_{s-1}} \cdot X.$$

Proof. (i) We use induction on s . If $s = 1$ then the result follows from Lemma 2.7(i) (noting that there the hypothesis was just on *least* elements, but here since M_0 is allowed to be ω or ω^* , we need the hypothesis on least and greatest). For the induction step, we assume the result for s , and write $A' = M_1^{n_1} \cdot M_2^{n_2} \cdots M_s^{n_s} \cdot X$ and $B' = M_1^{n_1} \cdot M_2^{n_2} \cdots M_s^{n_s} \cdot Y$. By induction hypothesis, $A' \equiv_{2(n_1+n_2+\dots+n_s)} B'$, and from Lemma 2.9 we deduce that $A \equiv_{2(n_0+n_1+\dots+n_s)} B$.

(ii) The basis case follows from Lemma 2.7(ii), and the induction step follows by considering $M_1^{n_1} \cdot M_2^{n_2} \cdots M_s^{n_s}$ and $M_1^{n_1} \cdot M_2^{n_2} \cdots M_s^{n_s} \cdot X$, which by induction hypothesis are $2(n_1 + n_2 + \dots + n_s) + 1$ -inequivalent, and again applying Lemma 2.9. ■

For the next result we require a slightly different strengthening of Lemma 2.7.

Lemma 3.3. *For any integers $n \geq m > 0$ and non-empty linear orders X and Y ,*

(i) *if neither X nor Y has a least element, then $\omega^m \cdot X \equiv_{2m+1} \omega^n \cdot Y$,*

(ii) *if neither X nor Y has a greatest or least element, then $\omega^m \cdot X \equiv_{2m+2} \omega^n \cdot Y$.*

Proof. (i) Let $A = \omega^m \cdot X$ and $B = \omega^n \cdot Y$, and write $A' = X$, $B' = \omega^{n-m} \cdot Y$. Then A' and B' are non-empty, and so $l(A', B') \geq 1$, and neither of them has a least element. So by Lemma 2.9(i), $l(A, B) \geq 2m + 1$.

(ii) With the same A' and B' , this time neither A' nor B' has a greatest or least, so $l(A', B') \geq 2$, and by Lemma 2.9(i) again, $l(A, B) \geq 2m + 2$. ■

We are now able to give a more general result about monomials, first comparing ones of equal lengths.

Theorem 3.4. *Let $A = M_0^{m_0} \cdot M_1^{m_1} \cdots M_{s-1}^{m_{s-1}} \neq B = M_0^{n_0} \cdot M_1^{n_1} \cdots M_{s-1}^{n_{s-1}}$ where m_i and n_i are non-zero finite numbers for every $i < s$ and M_0, M_1, \dots alternate between ω and ω^* and let t be the smallest i such that $m_i \neq n_i$. (Assume without loss of generality that $m_t < n_t$.)*

- (i) *If $t = s - 1$, then $l(A, B) = 2(m_0 + m_1 + \cdots + m_t)$,*
- (ii) *If $t = s - 2$, then $l(A, B) = 2(m_0 + m_1 + \cdots + m_t) + 1$,*
- (iii) *If $t < s - 2$, then $l(A, B) = 2(m_0 + m_1 + \cdots + m_t) + 2$.*

Proof. Throughout we assume without loss of generality that $M_0 = \omega$.

(i) We use induction. If $s = 1$ then the result follows from Lemma 2.7. Otherwise, if $s > 0$, write A as $\omega^{m_0} \cdot A'$, and B as $\omega^{m_0} \cdot B'$, where $A' = M_1^{m_1} \cdot M_2^{m_2} \cdots M_{s-1}^{m_{s-1}}$ and $B' = M_1^{n_1} \cdot M_2^{n_2} \cdots M_{s-1}^{n_{s-1}}$. By induction hypothesis, $l(A', B') = 2(m_1 + \cdots + m_{s-1})$. Since neither A' nor B' has a least element, by Lemma 2.9, $l(A, B) = 2(m_0 + m_1 + \cdots + m_{s-1})$.

(ii) We first show that $A \not\equiv_{2(m_0+m_1+\cdots+m_t)+2} B$. If $s \leq 2$, then as $s - 2$ exists, $s = 2$, and $A = \omega^{m_0} \cdot (\omega^*)^{m_1}$, $B = \omega^{n_0} \cdot (\omega^*)^{n_1}$ where $m_0 < n_0$. Thus $A = \omega^{m_0} \cdot A'$, $B = \omega^{m_0} \cdot B'$ where $A' = (\omega^*)^{m_1}$ and $B' = \omega^{n_0-m_0} \cdot (\omega^*)^{n_1}$. Since A' has a greatest and B' does not, $l(A', B') = 1$, so by Lemma 2.9(i), $l(A, B) = 2m_0 + 1$ (as neither A' nor B' has a least).

If $s \geq 3$, we use induction. Let $A' = M_1^{m_1} \cdot M_2^{m_2} \cdots M_{s-1}^{m_{s-1}}$ and $B' = M_1^{n_1} \cdot M_2^{n_2} \cdots M_{s-1}^{n_{s-1}}$, so that $A = \omega^{m_0} \cdot A'$ and $B = \omega^{m_0} \cdot B'$. By induction hypothesis, $l(A', B') = 2(m_1 + m_2 + \cdots + m_{s-2}) + 1$. Since A', B' do not have least elements, by Lemma 2.9(i) it follows that $l(A, B) = 2(m_0 + m_1 + \cdots + m_t) + 1$.

(iii) First consider the case where $t = 0$, and we start by showing that $A \not\equiv_{2m_0+3} B$. Let I choose the first point b_1 of an ω^{n_0} -block of B , and let a_1 be II's response. Then either a_1 is the first point in its ω^{m_0} -block, in which case, as $M_1 = \omega^*$, there is an immediately preceding block, so I can play $a_2 < a_1$ so that $[a_2, a_1] \cong \omega^{m_0}$, or if not, I can play $a_2 < a_1$ so that $[a_2, a_1] \cong \omega^k$ where $k < m_0$ (possibly $k = 0$). Let b_2 be II's reply. Then $b_2 < b_1$ and so $[b_2, b_1]$ is a multiple of ω^{n_0} . By Lemma 2.7(ii), $\omega^k \not\equiv_{2k+1} (b_2, b_1)$, so I can win in at most $2k + 1 \leq 2m_0 + 1$ more moves (if $k = 0$, then $(a_2, a_1) = \emptyset$, so I wins on the next move by playing between b_2 and b_1), meaning that he has used at most $2m_0 + 3$ moves in all.

The fact that $A \equiv_{2m_0+2} B$ follows at once from Lemma 3.3(ii).

Now consider the general case $t > 0$, and we use induction. We write $A = \omega^{m_0} \cdot A'$ and $B = \omega^{m_0} \cdot B'$, where $A' = M_1^{m_1} \cdots M_{s-1}^{m_{s-1}}$ and $B' = M_1^{n_1} \cdots M_{s-1}^{n_{s-1}}$. By induction hypothesis, $l(A', B') = 2(m_1 + \cdots + m_t) + 2$, and the result now follows by appeal to Lemma 2.9. ■

We extend the same result to the case of monomials of different lengths.

Theorem 3.5. *Let $A = M_0^{m_0} \cdot M_1^{m_1} \cdots M_{s-1}^{m_{s-1}}$ and $B = M_0^{n_0} \cdot M_1^{n_1} \cdots M_{t-1}^{n_{t-1}}$ where m_i and n_i are non-zero finite numbers for $i < s$, $j < t$, $s < t$, and M_0, M_1, \dots alternate between ω and ω^* and let u be the smallest i such that $m_i \neq n_i$ (or, if $m_i = n_i$ for all $i < s$ then we let $u = s$).*

- (i) *If $u = s$ or $s - 1$, then $l(A, B) = 2(m_0 + m_1 + \cdots + m_{s-2}) + 1$,*

- (ii) If $u = s - 2$, and $m_{s-2} < n_{s-2}$, then $l(A, B) = 2(m_0 + m_1 + \cdots + m_{s-3} + m_{s-2}) + 1$, and if $n_{s-2} < m_{s-2}$, then $l(A, B) = 2(m_0 + m_1 + \cdots + m_{s-3} + n_{s-2}) + 2$,
- (iii) If $u < s - 2$, then $l(A, B) = 2(m_0 + m_1 + \cdots + m_{u-1} + \min(m_u, n_u)) + 2$.

Proof. (i) First suppose that $s = 1$. Assume without loss of generality that $M_0 = \omega$. Thus $A = \omega^{m_0}$ and $B = \omega^{n_0} \cdot (\omega^*)^{n_1} \cdots$ where $m_0 = n_0$ if $u = 1$ and $m_0 \neq n_0$ if $u = 0$. By Lemma 2.9(ii), $l(A, B) = 1 = 2(m_0 + \cdots + m_{s-2}) + 1$.

If $s = 2$, then $A = \omega^{m_0} \cdot (\omega^*)^{m_1}$ and $B = \omega^{m_0} \cdot (\omega^*)^{n_1} \cdot \omega^{n_2} \cdots$ (where again $m_1 = n_1$ or $m_1 \neq n_1$ depending on the value of u). Here if we let $A' = (\omega^*)^{m_1}$ and $B' = (\omega^*)^{n_1} \cdot \omega^{n_2} \cdots$, then by the case $s = 1$ (with ω^* in place of ω), $l(A', B') = 1$. As neither A' nor B' has a least, by Lemma 2.9(i), $l(A, B) = 2m_0 + 1$, as desired.

Now let $s > 2$ and we use induction. Writing $A' = M_1^{m_1} \cdots M_{s-1}^{m_{s-1}}$, $B' = M_1^{n_1} \cdots M_{t-1}^{n_{t-1}}$, by the induction hypothesis, $l(A', B') = 2(m_1 + \cdots + m_{s-2}) + 1$, and by appeal to Lemma 2.9 again, $l(A, B) = 2(m_0 + m_1 + \cdots + m_{s-2}) + 1$.

(ii) If $s = 2$ then assuming $M_0 = \omega$, $A = \omega^{m_0}(\omega^*)^{m_1}$ and $B = \omega^{n_0}(\omega^*)^{n_1}\omega^{n_2} \cdots$. If $m_0 < n_0$ then we write $A = \omega^{m_0} \cdot A'$ and $B = \omega^{m_0} \cdot B'$, where neither $A' = (\omega^*)^{m_1}$ nor $B' = \omega^{n_0-m_0}(\omega^*)^{n_1}\omega^{n_2} \cdots$ has a least element. Clearly $l(A', B') = 1$, so by Lemma 2.9(i), $l(A, B) = 2m_0 + 1$. If $n_0 < m_0$ then $A = \omega^{n_0} \cdot A'$ and $B = \omega^{n_0} \cdot B'$, where $A' = \omega^{m_0-n_0}(\omega^*)^{m_1}$, $B' = (\omega^*)^{n_1}\omega^{n_2} \cdots$ again do not have least elements. This time, $l(A', B') = 2$, giving $l(A, B) = 2n_0 + 2$. For I can win in 3 moves by playing the first point of the final $\omega^{m_0-n_0}$ block in A' . Whatever $b_1 \in B'$ II plays has an immediate predecessor b_2 which I plays, and then I wins on the third move by playing in A' between a_2 and a_1 . Since however A' and B' have neither greatest nor least, $A' \equiv_2 B'$.

For general $s > 2$ we argue inductively, and write $A = \omega^{m_0} \cdot A'$, $B = \omega^{m_0} \cdot B'$. By induction hypothesis the result holds for A' and B' , and this lifts to A and B by using Lemma 2.9(i).

(iii) Here we have $A = M_0^{m_0} \cdots M_{u-1}^{m_{u-1}} M_u^{m_u} \cdots M_{s-1}^{m_{s-1}}$ and $B = M_0^{m_0} \cdots M_{u-1}^{m_{u-1}} M_u^{n_u} \cdots M_{t-1}^{n_{t-1}}$. First suppose that $u = 0$ and $M_0 = \omega$, so $A = \omega^{m_0} \cdots M_{s-1}^{m_{s-1}}$, $B = \omega^{n_0} \cdots M_{t-1}^{n_{t-1}}$, $m_0 < n_0$, $s > 2$. Player I plays the first point b_1 of an ω^{n_0} block in B . Then as in the proof of Theorem 3.4, whatever $a_1 \in A$ II plays, I can play $a_2 < a_1$ so that $[a_2, a_1] \cong \omega^k$ for some $k \leq m_0$ (in the same ω^{m_0} block as a_1 if it is not the first point of the block, or else in the immediately preceding block if it is the first). Then $[b_2, b_1]$ must be a multiple of ω^{n_0} , and as $m_0 < n_0$, by Lemma 2.7(ii), I can win in at most $2m_0 + 1$ further moves, making at most $2m_0 + 3$ in all.

A similar argument applies if $n_0 < m_0$, and the general case follows by induction using Lemma 2.9(i) as before. ■

Corollary 3.6. *If $A = M_0^{m_0} \cdot M_1^{m_1} \cdots M_{s-1}^{m_{s-1}}$ and $B = M_0^{n_0} \cdot M_1^{n_1} \cdots M_{t-1}^{n_{t-1}}$ where m_i, n_j for $i < s, j < t$ are non-zero natural numbers, then $A \equiv B$ if and only if $s = t$ and $m_i = n_i$ for all i .*

Proof. \implies It follows from Theorem 3.5 that $s = t$. Suppose that $m_j \neq n_j$ for some j , and let u be the least such. Theorem 3.4 gives a precise value for m that depends on u such that $A \equiv_m B$ but $A \not\equiv_{m+1} B$. Therefore $A \not\equiv B$ which is a contradiction. Hence $m_j = n_j$ for all j .

\Leftarrow Suppose that $s = t$ and $m_i = n_i$ for all $i < s$. Then $A \cong B$ and hence $A \equiv B$. ■

Corollary 3.6 tells us that the simple form of a finite product monomial, say $A = M_0^{n_0} \cdot M_1^{n_1} \cdots M_{s-1}^{n_{s-1}}$ with finite exponents, is A itself and it is unique. The following corollary tells us about the simple form of an ω -monomial product with finite exponents.

Corollary 3.7. *If $A = M_0^{m_0} \cdot M_1^{m_1} \cdots$ and $B = M_0^{n_0} \cdot M_1^{n_1} \cdots$ are ω -monomial products where m_i, n_i are non-zero natural numbers, then $A \equiv B$ if and only if $m_i = n_i$ for all i .*

Proof. \implies Suppose that $m_i \neq n_i$ for some i , and let t be the smallest such i . By following the same strategy as in Theorem 3.4, player I can win in $2(n_0 + n_1 + \cdots + n_t) + 3$ moves, which shows that $A \not\equiv B$.

\Leftarrow Conversely, if $m_i = n_i$ for all i , then $A \cong B$, so $A \equiv B$. \blacksquare

Corollary 3.8. *Let $A = M_0^{m_0} \cdot M_1^{m_1} \cdots$ and $B = M_0^{n_0} \cdot M_1^{n_1} \cdots N$ where N is any monomial. Then A is elementarily equivalent to B .*

Proof. By the proof of Theorem 3.4, for each n , $A \equiv_n B$, and hence $A \equiv B$. \blacksquare

We now consider monomials that involve infinite exponents.

Theorem 3.9. *If $A = M_0^{m_0} \cdot M_1^{m_1} \cdots M_{t-1}^{m_{t-1}} \cdot M_t^\lambda \cdot M_{t+1}^{m_{t+1}} \cdots$ where every m_i is a non-zero natural number and $\lambda \geq \omega$ and $B = M_0^{m_0} \cdot M_1^{m_1} \cdots M_{t-1}^{m_{t-1}} \cdot M_t^\lambda \cdot M_{t+1}^{m_{t+1}}$ where $n_{t+1} \geq 1$, then $A \equiv_n B$ for all n , so $A \equiv B$.*

Proof. By Theorem 3.2, for each k , $A \equiv_{2(m_0+m_1+\dots+m_{t-1}+k)} B$, and hence $A \equiv_n B$ for all n . \blacksquare

We remark that the same method shows that $M_0^{m_0} \cdot M_1^{m_1} \cdots M_{t-1}^{m_{t-1}} \cdot M_t^\lambda \cdot M_{t+1}^{m_{t+1}} \equiv M_0^{m_0} \cdot M_1^{m_1} \cdots M_{t-1}^{m_{t-1}} \cdot M_t^\mu \cdot M_{t+1}^{n_{t+1}}$.

By Corollary 3.8, the simple form of a monomial which has at least one term with infinite exponent is obtained by replacing the first such term by the one with exponent ω , replacing the next term if any by one with exponent 1, and removing all subsequent terms.

Example 3.10. (i) $\omega^\omega \cdot \omega^* \equiv \omega^\omega \cdot (\omega^*)^n$ for any non-zero ordinal n .

(ii) $\omega^\omega \cdot (\omega^*)^{n_1} \cdot \omega^{n_2} \equiv \omega^\omega \cdot \omega^*$.

(iii) $M_0^{n_0} \cdot M_1^{n_1} \cdots M_s^{n_s} \cdot \omega^\omega \cdot \omega^* \equiv M_0^{n_0} \cdot M_1^{n_1} \cdots M_s^{n_s} \cdot \omega^\omega \cdot (\omega^*)^{k_1} \cdot \omega^{k_2} \cdot (\omega^*)^{k_3} \cdots$

(iv) $M_0^{n_0} \cdot M_1^{n_1} \cdots M_s^{n_s} \cdot \omega^\omega \cdot \omega^* \equiv M_0^{n_0} \cdot M_1^{n_1} \cdots M_s^{n_s} \cdot \omega^\alpha \cdot \omega^*$ for any $\alpha \geq \omega$.

Theorem 3.11. *Any member of Σ has a unique simple form.*

Proof. This is a consequence of Theorems 3.4, 3.5, 3.9, and Corollary 3.8. \blacksquare

Theorem 3.12. *Any monomial product is elementarily equivalent some monomial product of rank at most $\omega + 1$.*

Proof. This also follows from 3.4, 3.9, 3.9, and 3.8, and since, as we saw earlier, ζ can only cause the rank of a monomial product to grow by at most 1 up to elementary equivalence and that no terms that come after ζ appear in the simple form of any given monomial. \blacksquare

From the above work, we draw the following conclusions.

Any $X \in \Sigma$ is elementarily equivalent to a linear ordering of rank at most $\omega + 1$. Hence for monomials involving ω^*, ω , and n , the bound on the rank of such monomials modulo elementary equivalence is the same as the bound on the rank of ordinals modulo elementary equivalence. The presence of ζ causes rank to grow by at most 1 up to elementary equivalence. In fact, the bound on the rank of members of the new class of monomials up to elementary equivalence is still $\omega + 1$.

Briefly considering the non-scattered case, we let Σ_q be the monomial class of countable linear orderings which allows η , the order-type of the rationals, to appear as a term, that is, Σ_q is a class of linear orderings that are products of $\omega^{r_i}, (\omega^*)^{s_i}, \zeta^{t_i}, \eta$ and n where s_i, k_i and t_i are non-zero ordinals. We note that any

$L \in \Sigma_q$ is either a scattered countable monomial (if η does not appear), a dense countable monomial or falls in between. In fact as in Corollary 2.6, if η occurs at all, we may suppose up to elementary equivalence that it occurs just once, and at the end, so that the monomial has the form $A \cdot \eta$, where A is scattered.

4. On sums of monomials

The next stage in building scattered orderings according to Definition 2.1 is to look at sums of monomials over $\omega, \omega^*, \mathbb{Z}$, or a finite set as index set. The general case is too complicated for our current techniques, so we just give a few illustrative special cases. We assume non-redundancy of the expression as in the following definition.

Definition 4.1. *A monomial sum, W , is a linear ordering which is a sum of monomials over an index set $\omega, \omega^*, \mathbb{Z}$, or $m \in \omega$, such that if M and N are consecutive terms of W , then $M + N \neq M, N$.*

Each monomial occurring in a monomial sum is a *term*. First we look at the cases where the terms are just powers of ω or ω^* , and the index set is \mathbb{Z} . We remark on a special case, relevant for what follows, in which the terms on the right are eventually all powers ω^{n_i} of ω , from the k th on, say. The definition of ‘monomial sum’ entails that $n_k \geq n_{k+1} \geq \dots$, so this sequence is eventually constant, from the l th term on, say. Then $\omega^{n_l} + \omega^{n_{l+1}} + \dots = \omega^{n_l} \cdot \omega = \omega^{n_l+1}$. The \mathbb{Z} -indexed sum thus *could* be replaced by an ω^* -indexed one (though we do not do this, as we are seeking to compare similar sums of non-trivial terms). If however beyond the k th, all terms are powers of ω^* , the powers could increase. Dual remarks apply to behaviour on the left, with the roles of ω and ω^* interchanged.

Lemma 4.2. *If $A = \sum\{t_i : i \in \mathbb{Z}\}$ and $B = \sum\{u_i : i \in \mathbb{Z}\}$ are monomial sums, and there are natural numbers m_i and n_i such that for each i , $t_i = \omega^{m_i}$ or $(\omega^*)^{m_i}$, and u_i is either ω^{n_i} or $(\omega^*)^{n_i}$, then $A \cong B$ if and only if there is an automorphism θ of \mathbb{Z} such that for all i , $u_i = t_{\theta i}$.*

Proof. Automorphisms of \mathbb{Z} are of course just translations. Clearly if there is an automorphism of \mathbb{Z} as stated, then $A \cong B$. Conversely, suppose that $\varphi : A \rightarrow B$ is an isomorphism. We shall show that for each i , $\varphi(t_i) = u_j$ for some j , and the result then follows on letting $\theta(i) = j$.

First suppose that t_i is a singleton, i.e. $m_i = 0$, and let j be such that $\varphi(t_i) \subseteq u_j$. We show that u_j is also a singleton, which establishes that $\varphi(t_i) = u_j$. If this does not hold, then $\varphi(t_i)$ must be a proper subset of u_j , so $n_j \neq 0$. Assume without loss of generality that $u_j = \omega^{n_j}$. Not all t_k for $k \geq i$ can be singletons. For then they would all have to map into u_j under φ , and u_{j+1} would be disjoint from the image of φ . Let $k > i$ be least such that t_k is not a singleton. Then $t_k = (\omega^*)^{m_k}$ with $m_k > 0$ (as otherwise, the singletons would be ‘absorbed’ into t_k , contrary to the definition of ‘monomial sum’). But since φ is an isomorphism, and the points of B greater than $\varphi(t_i)$ begin with an ω -sequence, this is impossible.

Next assume that $m_i \neq 0$. Now $\varphi(t_i)$ intersects a finite non-empty set J_i of u_j s. This J_i is minimal such that $\varphi(t_i) \subseteq \bigcup_{j \in J_i} u_j$. Suppose without loss of generality that $t_i = \omega^{m_i}$, and let j and k be the least and greatest of J_i respectively. Then each $\varphi(t_i) \cap u_r$ is well-ordered. If $j < r < k$, $u_r \subseteq \varphi(t_i)$, and so $u_r = \omega^{n_r}$. At the left hand end point, $I = \varphi(t_i) \cap u_j$ is a final segment of u_j , so either $u_j = \omega^{n_j}$, in which case $\varphi(t_i) \cap u_j \cong \omega^{n_j}$ or $u_j = (\omega^*)^{n_j}$ and $\varphi(t_i) \cap u_j$ is finite. At the right hand end point, $F = \varphi(t_i) \cap u_k$ is an infinite initial segment of u_k isomorphic to ω^{m_i} , so that $u_k = \omega^{n_k}$ and $m_i \leq n_k$. Thus $\varphi(t_i) = I \cup \omega^{n_{j+1}} \cup \dots \cup \omega^{n_{k-1}} \cup F \cong$

$\omega^{n_j} + \omega^{n_{j+1}} + \dots + \omega^{n_{k-1}} + \omega^{m_i}$, or $I + \omega^{n_{j+1}} + \dots + \omega^{n_{k-1}} + \omega^{m_i}$ where I is finite. In each case, this is isomorphic to ω^{m_i} , so by definition of B a monomial sum, either $j = k$, or $k = j + 1$ and I is finite. To sum up, there are a finite set I and an initial segment F of ω^{n_k} isomorphic to ω^{m_i} such that $\varphi(t_i) = I \cup F$. Applying the same argument to $\varphi^{-1}u_k$ (noting that u_k cannot be a singleton) we see that actually $m_i = n_k$, and hence $\varphi(t_i) = u_j$ as required. ■

Lemma 4.3. *If α, β , and γ are ordinals, and $m > 1$, then*

- (i) γ infinite $\Rightarrow \alpha \not\equiv_4 \beta + \omega^m + \gamma^*$,
- (ii) β infinite $\Rightarrow \alpha \not\equiv_4 \beta + (\omega^*)^m + \gamma^*$.

Proof. We treat (i) and (ii) simultaneously. Let us write $\alpha = \lambda + k$ where λ is a limit ordinal or zero and k is finite. If $k = 0$, I plays the greatest point of γ^* and wins in 2 moves. If $k > 0$, I plays the first point a_1 of the final k -block of α . If II replies with b_1 in γ^* , (in (i)) or $(\omega^*)^m + \gamma^*$ (in (ii)), and b_1 is not least in this set, I plays its predecessor b_2 , and as a_1 has no predecessor, wins in 3 moves. If b_1 is the least in γ^* (this case does not arise in (ii)), I plays the ω th point of γ^* from the right (which may equal b_1). Note that this has no successor. Then if a_2 is II's reply, either a_2 is the maximum of α , in which case I wins on the next move, or else I can play its successor, and wins in 4 moves.

Otherwise, $b_1 \in \beta + \omega^m$ (in (i)), or β (in (ii)). Now in case (i) I plays some $b_2 > b_1$ in $\beta + \omega^m$ with no predecessor, possible since $m > 1$. Whatever $a_2 > a_1$ I plays in α has no predecessor, so I wins in 4 moves. In case (ii), player I instead plays some b_2 in the $(\omega^*)^m$ block with no successor, and whatever $a_2 > a_1$ I plays is either greatest in α or has a successor, so I wins in 2 more moves. ■

For technical reasons, which will be explained, we focus of ω -sums. Before formulating our main result on these, or indeed on \mathbb{Z} -sums, we have to deal with the fact that by Lemma 2.5, $\omega + \omega^* + \omega \equiv \omega$, which means that we have to make a further restriction in the monomial sums which arise in our main theorem. We summarize these in the following result.

Lemma 4.4. *If $k, l, m, n \geq 1$, then $\omega^k + (\omega^*)^l + \omega^m \equiv \omega^n \leftrightarrow k = l = 1$ and $m = n$. More precisely, if $m \neq n$ then $\omega^k + (\omega^*)^l + \omega^m \not\equiv_{2 \min(m,n)+2} \omega^n$, if $l > 1$ then $\omega^k + (\omega^*)^l + \omega^m \not\equiv_3 \omega^n$, and if $l = 1, m = n$, and $k > 1$, then $\omega^k + (\omega^*)^l + \omega^m \not\equiv_6 \omega^n$.*

Proof. First we note that if $m \neq n$ then player I can win by playing the least point a_1 of the ω^m block. Whatever point b_1 of ω^n player II plays, then by Lemma 1.3, player I wins on the right in at most $2 \min(m, n) + 1$ more moves, since $A^{>a_1} \cong \omega^m$ and $B^{>b_1} \cong \omega^n$. So from now on assume that $m = n$. If $l > 1$ then player I plays the ω th point a_1 of $(\omega^*)^l$ from the right, noting that this has no immediate successor. Whatever $b_1 \in \omega^n$ player II plays, b_1 has a successor b_2 , which I now plays, and whatever $a_2 > a_1$ II plays in $\omega^k + (\omega^*)^l + \omega^m$, I wins (in 3 moves) by playing between a_1 and a_2 . So from now on we may suppose that $l = 1$ and $m = n$.

If $k > 1$, then player I plays the last point a_1 in the copy of ω^* . Let II's reply be $b_1 = \omega^r p_r + \dots + \omega p_1 + p_0 \in \omega^n$. Since $A^{>a_1} \cong B^{>b_1} (\cong \omega^n)$ we concentrate on the left, where $A^{<a_1} \cong \omega^k + \omega^*$ and for some coefficients $p_i < \omega$, for $i \leq r$, where $r < m$, $B^{<b_1} \cong \omega^r p_r + \dots + \omega p_1 + p_0$. If $p_0 = 0$ then player I wins at once since $A^{<a_1}$ has a greatest but $B^{<b_1}$ does not, so we now suppose that $p_0 \neq 0$. If $p_r = \dots = p_2 = p_1 = 0$ then player I plays $a_2 = \omega$ in $A^{<a_1}$. This has no immediate predecessor, but every point of $B^{<b_1}$ does have (or is minimal), and so again I wins quickly. Otherwise there is a least $i > 0$ such that $p_i \neq 0$. Player I plays the first point b_2 of the last ω^i block in $B^{<b_1}$. Suppose that Player II plays a_2 . If $a_2 \in \omega^*$, then Player I plays the least point b_3 of the final p_0 block of (b_2, b_1) and wins since b_3 has no immediate predecessor, but any move that II can make *does*. Otherwise,

$a_2 \in \omega^k$, so $(a_2, a_1) \cong \omega^k + \omega^*$ and $(b_2, b_1) \cong \omega^i + p_0$. Player I plays the least point b_3 of the final p_0 block of (b_2, b_1) . Once more, if II plays in ω^* , I wins at once. If II plays $a_3 \in \omega^k$, then $(a_3, a_1) \cong \omega^k + \omega^*$ and (b_3, b_1) is finite, and again I wins by playing the ω th point of (a_3, a_1) .

Finally observe that $\omega + \omega^* + \omega^m = \omega + \omega^* + \omega + \omega^m \equiv \omega + \omega^m = \omega^m$. \blacksquare

The above result explains the reason for the restriction taken on monomial sums. Let us say that a monomial sum is *special* if no three consecutive terms are of the form ω , ω^* , ω^m or $(\omega^*)^m$, ω , ω^* for any $m \geq 1$. Note that we may reduce any monomial sum in the previous sense to an elementarily equivalent special one using Lemma 4.4. The only seeming exception would be if the terms eventually alternate between ω and ω^* on the right, or on the left; but then the right is $\omega + \zeta \cdot \omega \equiv \omega$ by Lemma 2.5 (and similarly for such behaviour on the left), so we can reduce to a monomial over bounded domain (which we don't actually treat, but we could do so).

Theorem 4.5. *If $A = \sum\{t_i : i \in \omega\}$ and $B = \sum\{u_i : i \in \omega\}$ are special monomial sums, and there are natural numbers m_i and n_i such that for each i , $t_i = \omega^{m_i}$ or $(\omega^*)^{m_i}$, and u_i is either ω^{n_i} or $(\omega^*)^{n_i}$, then $A \cong B$ if and only if $A \equiv B$.*

Proof. Clearly $A \cong B \Rightarrow A \equiv B$, so we concentrate on showing that $A \not\cong B \Rightarrow A \not\equiv B$. We therefore assume that A and B are not isomorphic, and aim to show that they are not elementarily equivalent, that is, that player I has a winning strategy in some finite Ehrenfeucht-Fraïssé game on A and B . For ease we suppose that whenever m_i or n_i equals 0, then the corresponding term is a power of ω rather than ω^* . Since $A \not\cong B$, there is a least N such that $t_N \not\cong u_N$. We shall describe an Ehrenfeucht-Fraïssé game on a (finite) number of moves on A and B , whose precise value (depending on N) can be read off from the proof, and which is winning for player I.

Case 1: $A = \alpha$ and $B = \beta$ are both ordinals. Since $A \not\cong B$, $\alpha \not\cong \beta$. Since $m_0 \geq m_1 \geq m_2 \geq \dots$, this sequence is eventually constant, and it follows that $\alpha < \omega^\omega$, and similarly $\beta < \omega^\omega$. Hence, by Lemma 1.4, player I has a winning strategy.

Case 2: One of A and B , A say, is an ordinal, the other not. If some member b_1 of B has no immediate successor, then player I plays it on his first move. Whatever $a_1 \in A$ player II plays, a_1 has a successor a_2 , which I now plays, and whatever $b_2 > b_1$ II plays in B , I can play b_3 between b_1 and b_2 , and wins on the third move. If however, every member of B has a successor, then any i such that $u_i = (\omega^*)^{n_i}$ must have $n_i = 1$, and furthermore this cannot happen for consecutive terms. If such i is 0, then A has a least member and B does not, so I wins in 2 moves by playing the least member of A on his first move, following it by a member of B less than whatever II plays. If such i is non-zero, then we have $u_{i-1} + u_i + u_{i+1} = \omega^{n_{i-1}} + \omega^* + \omega^{n_{i+1}}$, and as every member of B has a successor, $n_{i-1} > 0$, and as it is special, $n_{i-1} > 1$. On his first two moves, player I plays the first point b_1 of u_{i-1} and the last point b_2 of u_i . If a_1 and a_2 are II's responses in A , then (a_1, a_2) is well-ordered, so I wins in 4 more moves by Lemma 4.3.

From now on we therefore assume that neither A nor B is an ordinal. Let α, β be ordinals (possibly 0) such that for some i, j , $\bigcup_{k < i} t_k \cong \alpha$, $\bigcup_{k < j} u_k \cong \beta$, $t_i = (\omega^*)^{m_i}$, $u_j \cong (\omega^*)^{n_j}$.

Case 3: In the notation just introduced, $\alpha \neq \beta$ (which is equivalent to $N < \max(i, j)$). By interchanging A and B if necessary, suppose that $\alpha > \beta$. Then by Lemma 1.4, $\alpha \not\equiv_{2m_0+i} \beta$. Using the winning strategy provided by the proof of that lemma, I plays the first points a_1, a_2, \dots, a_i of t_0, t_1, \dots, t_{i-1} respectively, and

let II's replies be b_1, b_2, \dots, b_i . If the game continued as far as this, then $\beta \geq \alpha$, contrary to assumption. Hence there is a least k such that $b_k \in \bigcup_{l \geq j} u_l$. Note that $a_k \leq \min(t_{i-1})$. Now I stops playing the above sequence, and instead plays $b_{k+1} \in u_j$ so that $b_{k+1} < b_k$. Whatever $a_{k+1} < a_k$ II now plays, $A^{<a_{k+1}}$ is an ordinal and $B^{<b_{k+1}} \cong \beta + (\omega^*)^{n_j}$, so I can win in 4 more moves by appealing to Lemma 4.3, except in one case, which is when $u_{j-1} = \omega$ and $u_j = \omega^*$ (i.e. $n_j = 1$). But now, as B is a special monomial sum, u_{j+1} must equal $(\omega^*)^{n_{j+1}}$. If b_{k+1} is the maximum of u_j , it therefore has no successor, so I wins in two more moves by playing the successor a_{k+2} of a_{k+1} . If b_{k+1} is not the maximum of u_j , then I plays the maximum b_{k+2} of u_j on his next move. As this is the first point of B having no immediate successor, to avoid losing quickly, player II must play the first point a_{k+2} of A , if any, having no successor (which could only be the greatest point of some ω^* -block which is followed by an $(\omega^*)^{m_i}$ -block), so $a_{k+2} \in \bigcup_{l \geq i} t_l$. Now (b_{k+1}, b_{k+2}) is finite, and I plays the first point a_{k+3} of t_{i-1} , which lies in (a_{k+1}, a_{k+2}) since $a_k \leq \min(t_{i-1})$. This has no immediate predecessor, and therefore he wins on the next move.

Case 4: $\alpha = \beta$ (which implies that $N \geq i$), and for $i \leq k \leq N$, $t_k = (\omega^*)^{m_k}$ and $u_k = (\omega^*)^{n_k}$. Player I plays the greatest point a_1 of t_N . If II replies with $b_1 \in \alpha$, then I can win by appeal to Lemma 4.3. Note that this is inapplicable only in the case in which $t_{i-1} = \omega$ and $t_i = \omega^*$; but then, as A is a special monomial sum, t_{i+1} must equal $(\omega^*)^{m_{i+1}}$, and hence the greatest point a_2 of t_i has no successor, and I wins by playing it on his second move. If $b_1 \in \bigcup_{i \leq k \leq N} u_i$ then I restricts the play to $A^{<a_1} \cong \alpha + \gamma^*$ and $B^{<b_1} \cong \alpha + \delta^*$ for non-isomorphic ordinals γ and $\delta < \omega^\omega$. (If $b_1 \in \bigcup_{i < k < N} u_i$ then $\delta < \gamma$; if $b_1 \in u_N$ then $\gamma \neq \delta$ by definition of N .) He plays on $\gamma^* \cup \delta^*$ using a winning strategy provided by Lemma 1.4 until II plays in α (in A or B), and then he again wins using Lemma 4.3. If $b_1 \in \bigcup_{l > N} u_l$ then I plays the greatest point b_2 of u_N , and the play proceeds in the way just described with the roles of A and B reversed.

Case 5: $\alpha = \beta$, and for $i \leq k \leq N$, $t_k = (\omega^*)^{m_k}$ but $u_N = \omega^{n_N}$ (which implies that for $i \leq k < N$, $u_k = (\omega^*)^{m_k}$). Player I again plays the greatest point a_1 of t_N , and if II's reply b_1 is in $\bigcup_{l < N} u_l$, the same argument as in the previous case applies. In fact this is still true even if b_1 lies in the first ω points of u_N , since then $B^{<b_1} \cong \bigcup_{l < N} u_l$. So we now assume that $b_1 \in \bigcup_{l \geq N} u_l$, and $B^{<b_1}$ has an initial infinite well-ordered set beyond $\bigcup_{l < N} u_l$.

Player I now plays the least point b_2 of u_N , and by the arguments given in Case 4, player II must play some point a_2 of t_{N-1} (since these are precisely the points for which $A^{<a_2} \cong B^{<b_2}$). If there is a least point b_3 of B greater than the first ω points of u_N , player I plays this on his third move. Then b_3 has no predecessor, but whatever $a_3 \in (a_2, a_1)$ II now plays *does*, so I wins in two more moves. If there is no least point of B greater than the first ω points of u_N , then $u_N = \omega$, and $u_{N+1} = (\omega^*)^{n_{N+1}}$, and as the monomial sums are special, $n_{N+1} > 1$. Player I plays some $b_3 \in u_{N+1}$ less than b_1 . Now $(b_2, b_3) \cong \omega + (\omega^*)^{n_{N+1}}$ and whatever a_3 II plays, (a_2, a_3) is a reversed ordinal, so player I wins in 4 more moves by Lemma 4.3 (for the reversed ordering).

Case 6: $\alpha = \beta$, and for some $k \leq N$, $t_l = u_l = (\omega^*)^{m_l}$ for $i \leq l < k$, and $t_k = \omega^{m_k}$.

Thus $\bigcup_{l < i} t_l \cong \bigcup_{l < i} u_l \cong \alpha$, and $\bigcup_{i \leq l < k} t_l = \bigcup_{i \leq l < k} u_l \cong \gamma^*$, for some ordinal γ .

Player I plays the greatest point a_1 of t_{k-1} . If II's reply lies in $\bigcup_{l < k-1} u_l$ then we may argue as in Case 4 that I can win. If $b_1 \in u_{k-1}$, but is not among the final ω^* points, then I plays the greatest point b_2 of u_{k-1} . Let a_2 be II's reply. If (a_1, a_2) is well-ordered, either b_1 has no successor or I can play $b_3 \in (b_1, b_2)$ having no successor, and wins in 2 more moves. Otherwise, if possible, I plays $a_3 \in (a_1, a_2)$

with no predecessor and wins in 2 more moves. If this is not possible, then $m_k = 1$ and $t_{k+1} = (\omega^*)^{m_{k+1}}$, in which case I plays some point a_3 of t_{k+1} in (a_1, a_2) . Since the monomials are special, $m_{k+1} > 1$ and I wins by appeal to Lemma 4.3.

Next suppose that $b_1 \in \bigcup_{l \geq k} u_l$. Now player I plays the greatest point b_2 of u_{k-1} and the whole argument is rerun with A and B interchanged.

The only case remaining is that b_1 lies in the final ω^* points of u_{k-1} (or in the reversed argument, that a_2 lies in the final ω^* points of t_{k-1}). If $u_k = (\omega^*)^{n_k}$, player I plays the greatest point b_2 of u_k . If a_2 is II's reply, I plays $a_3 \in (a_1, a_2)$ if possible having no predecessor, and wins in 2 more moves. If this is not possible, then $m_k = 1$ and $t_{k+1} = (\omega^*)^{m_{k+1}}$, and as A is a special monomial sum, $m_{k+1} > 1$. Player I plays the greatest point a_3 of t_{k+1} , and wins using Lemma 4.3.

Otherwise, $u_k = \omega^{n_k}$. Now we see that $A^{<a_1} \cong B^{<b_1} \cong \alpha + \gamma^*$, and $A^{>a_1} \cong \bigcup_{l \geq k} t_l$, and $B^{>b_1} \cong \bigcup_{l \geq k} u_l$. We may now argue inductively, since there are fewer blocks between t_k, u_k and t_N, u_N than from t_0, u_0 . \blacksquare

The size of the class of such structures up to elementary equivalence is therefore 2^{\aleph_0} . We remark that the result we would really like is just as for Theorem 4.5, but with ω -sums replaced by \mathbb{Z} -sums. The problem in proving the result in this case is as follows. Assuming that $A \not\cong B$, player I endeavours to find a finite Ehrenfeucht-Fraïssé game on A and B that he can win. He plays some $a_1 \in A$ for instance, and player II replies with $b_1 \in B$. We now know that $A^{>a_1} \not\cong B^{>b_1}$ or $A^{<a_1} \not\cong B^{<b_1}$, suppose the former. This essentially reduces the problem to an ω -sum, so should be amenable to the methods presented in Theorem 4.5. The trouble is however, that the number of moves required by player I may depend on the choice of b_1 by player II, and player I may not be able to predict this number in advance, which it is essential that he can do.

Next we consider sums of more general terms, however only in a special case, namely of two terms $A = A_1 + A_2$ and $B = B_1 + B_2$, where A_1, A_2 start differently (one with ω , the other with ω^*) and B_1, B_2 start in the same ways respectively. Generally we'd expect that $l(A, B) = \min(l(A_1, B_1), l(A_2, B_2))$, since we could imagine that the players will play either on the left, or on the right, but this isn't necessarily true. For an easy example consider $A_1 = \omega^*$, $B_1 = (\omega^*)^2$, $A_2 = B_2 = \omega \cdot \omega^*$, where $l(A_1, B_1) = 2$, $l(A_2, B_2) = \infty$, but $l(A, B) = 3$ (see below). A precise list of the ways in which the expected result fails is included in the following result.

Theorem 4.6. *Let $A = A_1 + A_2$ and $B = B_1 + B_2$ where $A_1 = M_0^{m_0} \cdot M_1^{m_1} \cdots M_{s-1}^{m_{s-1}}$, $A_2 = N_0^{n_0} \cdot N_1^{n_1} \cdots N_{t-1}^{n_{t-1}}$ and $B_1 = M_0^{p_0} \cdot M_1^{p_1} \cdots M_{u-1}^{p_{u-1}}$, $B_2 = N_0^{q_0} \cdot N_1^{q_1} \cdots N_{v-1}^{q_{v-1}}$ be sums of two monomials where*

- (i) *the M_i alternate between ω and ω^* , and so do the N_i and,*
- (ii) *$M_0 = \omega \Leftrightarrow N_0 = \omega^*$.*

Then $l(A, B) = \min(l(A_1, B_1), l(A_2, B_2))$, except when $M_0 = \omega^$, in the following cases:*

- $s = u = 1, t, v \geq 2$, and $1 = \min(m_0, p_0)$, $m_0 \neq p_0$,
- $t = v = 1, s, u \geq 2$, and $1 = \min(n_0, q_0)$, $n_0 \neq q_0$,
- $s = 1, u \geq 2$, and $t \geq 2$ or $t = 1$ and $m_0 \geq 2$,
- $u = 1, s \geq 2$, and $v \geq 2$ or $v = 1$ and $p_0 \geq 2$,
- $t = 1, v \geq 2$, and $s \geq 2$ or $s = 1$ and $n_0 \geq 2$,
- $v = 1, t \geq 2$, and $u \geq 2$ or $u = 1$ and $q_0 \geq 2$, *in which case $l(A, B) = 3$*
- $s = 1, u \geq 2$, and $t = m_0 = 1$,
- $u = 1, s \geq 2$, and $v = p_0 = 1$,
- $t = 1, v \geq 2$, and $s = n_0 = 1$,
- $v = 1, t \geq 2$, and $u = q_0 = 1$, *in which case $l(A, B) = 2$.*

Proof. First we deal with the exceptional cases. We remark that in all of these, the formula $l(A, B) = \min(l(A_1, B_1), l(A_2, B_2))$ is invalid. In the first, for instance, $l(A_1, B_1) = 2$, and $l(A_2, B_2) \geq 2$, which gives $\min(l(A_1, B_1), l(A_2, B_2)) = 2$, and if $s = 1$ and $u \geq 2$, then $l(A_1, B_1) = 1$.

We first consider $s = u = 1$, $A_1 = \omega^*$, $B_1 = (\omega^*)^{p_0}$, $p_0 > 1$, and $t, v \geq 2$, so that A_2, B_2 have no least. For a winning strategy in $G_4(A, B)$, player I plays the last point b_1 of the last but one copy of ω^* in $(\omega^*)^{p_0}$, so that $B_1^{>b_1} \cong \omega^*$. If II plays $a_1 \in A$ which has a successor a_2 , then II plays a_2 on his second move, and whatever $b_2 > b_1$ in B II plays, as b_1 has no immediate successor, I can play b_3 between b_1 and b_2 and win on the third move. Since every member of A_2 has a successor, the only other option is that II plays the greatest point a_1 of A_1 . Now I plays the greatest point b_2 of B_1 , and I must respond in A_2 , and I wins in 2 more moves, making 4 in all (we have here used the fact that B_2 has no least). Next we have to see that II has a winning strategy in $G_3(A, B)$. For this we observe that if two linear orders each have at least 3 elements, and one has a minimum if and only if the other does, and similarly for maxima, then they are 2-equivalent. Note that player II can play in such a way that if the first two moves are both on the left, then for some $k \in \omega$ and λ which is a limit ordinal or 0, a_1 is the k th point from the right in A_1 and b_1 is the $(\lambda + k)$ th point from the right in B_1 . The remark just made, together with the assumption that A_2, B_2 do not have least elements (since they clearly do not have greatest either) shows that $A^{<a_1} \equiv_2 B^{<b_1}$ and $A^{>a_1} \equiv_2 B^{>b_1}$ as required. Similarly, if the play takes place on the right, II can ensure that a_1 is a successor if and only if b_1 is a successor, which is sufficient to guarantee the truth of the same condition.

The case of $t = v = 1$, $s, u \geq 2$, and $1 = \min(n_0, q_0)$, $n_0 \neq q_0$ is handled similarly (it is just obtained by reversing the ordering).

For the remaining ‘exceptional’ cases, it suffices to consider that of $s = 1$ and $u \geq 2$, subdivided into $t \geq 2$, $t = 1$ and $m_0 \geq 2$, and $t = m_0 = 1$. (The others are essentially the same, bearing in mind that the right hand terms begin with ω instead of ω^* , which is compensated for by the fact that we are now reading ‘in the other direction’.)

For $s = 1$, $u \geq 2$, $t \geq 2$, we have $A_1 = (\omega^*)^{m_0}$, $A_2 = \omega^{n_0} \cdot (\omega^*)^{n_1} \dots N_{t-1}^{n_{t-1}}$. We see that I can win in 4 moves. On his first move he plays the greatest point of A_1 . If II plays $b_1 \in B_2$, then I plays its successor on the next move, and wins in 3 moves. If II plays $b_1 \in B_1$, I plays $b_2 > b_1$ in B_1 which has no successor. Whatever move a_2 II plays, it lies in A_2 , so has a successor, which I can play on his 3rd move, and I wins in 4 moves in all. Now we have to show that II can win the 3-move game. We remark that two infinite linear orders are 2-equivalent if and only one has a least element if and only if the other does, and one has a greatest element if and only if the other does. Since neither A nor B has a greatest or least element, we just have to note that both orders have the same 3 options for $A^{<a}$ and $A^{>a}$ occurring, namely there are $a_i \in A$ such that $A^{<a_1}$ has a greatest and $A^{>a_1}$ has a least (a_i is an ‘inner’ point of a copy of ω or ω^*), $A^{<a_2}$ has a greatest and $A^{>a_2}$ has no least, and $A^{<a_3}$ has no greatest and $A^{>a_3}$ has a least, and similarly for B . Thus II can respond to whatever I plays on his first move so that if a and b are the first moves played, then $A^{<a} \equiv_2 B^{<b}$ and $A^{>a} \equiv_2 B^{>b}$, so II can win.

Next consider $s = 1$, $u \geq 2$, $t = 1$ and $m_0 \geq 2$. Thus $A_1 = (\omega^*)^{m_0}$, $A_2 = \omega^{n_0}$. To win in 4 moves, I starts by playing the ω th point a_1 of A_1 from the right. If II plays in B_2 , I plays its successor and wins in 3 moves. If II plays $b_1 \in B_1$, I plays $b_2 > b_1$ in B_1 with no successor. Whichever $a_2 > a_1$ II plays, it has a successor, so I can win in 2 more moves, making 4 in all. A similar argument to the previous paragraph shows that II can win in 3 moves.

If $s = 1$, $u \geq 2$, and $t = m_0 = 1$, then $A_1 = \omega^*$, $A_2 = \omega^{m_0}$, so every point of A has a successor. I can win in 3 moves by playing a point b_1 of B_1 with no successor (the greatest point of a copy of $(\omega^*)^{p_0}$ for instance); now any response a_1 by II has a successor, so I can win in 2 more moves. Here we have to see that II can win in 2 moves. This is immediate since A and B have no greatest or least.

Now let us move on to the general (non-exceptional) case.

Let us write m for $\min(l(A_1, B_1), l(A_2, B_2))$, and suppose without loss of generality that $m = l(A_1, B_1) = l(M_0^{m_0} \cdot M_1^{m_1} \cdots M_{s-1}^{m_{s-1}}, M_0^{p_0} \cdot M_1^{p_1} \cdots M_{u-1}^{p_{u-1}})$ is finite. In m moves, Player II has strategies for both games $G(M_0^{m_0} \cdot M_1^{m_1} \cdots M_{s-1}^{m_{s-1}}, M_0^{p_0} \cdot M_1^{p_1} \cdots M_{u-1}^{p_{u-1}})$ and $G(N_0^{n_0} \cdot N_1^{n_1} \cdots N_{t-1}^{n_{t-1}}, N_0^{q_0} \cdot N_1^{q_1} \cdots N_{v-1}^{q_{v-1}})$, and so he wins by using whichever one is required, depending on which side I's moves lie.

A rather more complicated argument is required to show that player I can win in at most $m + 1$ moves, as it isn't clear how he can ensure that the other player plays on the same side as him (which is the left, in view of our assumption on m). The best hope is that player I can 'force' the play to take place on the left. If II strays to the right in the early stages, then I can ensure a quick win, and otherwise, his strategy will ensure that the play is to the left of already played moves, unlike in the exceptional cases. We follow the proofs of Theorems 3.4 and 3.5, which compare monomials of the same and different lengths respectively. In the induction steps, we can ensure that this happens without too much trouble. Suppose that $A_1 = M_0^{m_0} \cdot A'$, and $B_1 = M_0^{m_0} \cdot B'$, and assume that I has a winning strategy σ for the game on A' and B' in l moves. We show that he can convert this into a winning strategy on the game between A and B in $2m_0 + l$ moves. Let I play in $A_1 \cup B_1$ according to σ on the copies of $M_0^{m_0}$, playing the first or last point depending on whether $M_0 = \omega$ or ω^* . If ever in the first l moves, II plays in $A_2 \cup B_2$, then I can win in 2 more moves by playing the predecessor or successor of II's last move in $A_2 \cup B_2$, in the two cases, and then II must respond with a point *not* adjacent to I's last move in $A_1 \cup B_1$, and I wins by playing in between the two in $A_1 \cup B_1$. Otherwise, as σ is winning in the game on A' and B' , at some stage by the l th, player II plays in the same copy of $M_0^{m_0}$ more than once. By the argument given in the proof of Lemma 2.9(i) for instance, player I can win in at most $2m_0$ more moves, and this is because he forces the play to lie between points which have already been played *on the left*.

This leaves us to deal with various 'basis cases'. Some of these also follow by the same argument, since they use variants of the 2-phase lemma, and one or two require separate treatment.

There are two cases arising in the basis of Theorem 3.4(i), depending on whether $M_0 = \omega$ or ω^* . In the first of these, where $A_1 = \omega^{m_0}$, $B_1 = \omega^{p_0}$, $m_0 < p_0$, player I plays the ω^{m_0} th point b_1 of B_1 . If II plays in A_2 , then I wins in at most 2 more moves, and $3 \leq 2m_0 + 1$. Otherwise, $a_1 \in A_1$, and I wins as usual, forcing the play to the left of the initial moves.

In the second case, $A_1 = (\omega^*)^{m_0}$, $B_1 = (\omega^*)^{p_0}$, $m_0 < p_0$. First note that as $l(A_1, B_1) = 2m_0$ and $l(A_1, B_1) \leq l(A_2, B_2)$, A_2 has a least if and only if B_2 has a least (and they both have no greatest). For if one of them has a least and the other does not, then $l(A_2, B_2) = 1$. If $m_0 \geq 2$, I plays the ω^{m_0} th point b_1 of B_1 from the right. If II's move a_1 lies in A_2 , then I plays its successor and wins in ≤ 3 moves. If $a_1 \in A_1$ is not the greatest in A_1 , I plays $a_2 > a_1$ in A_1 so that for some $k < m_0$, $(a_1, a_2] \cong (\omega^*)^k$. If II replies in B_1 , then the game continues in $A_1 \cup B_1$ on $(\omega^*)^{m_0}$ and $(\omega^*)^{m_0} \cdot 2$, so I wins in $2k + 1$ more moves, $\leq 2m_0 - 1$. If II replies in B_2 then I wins in 2 more moves, and $4 \leq 2m_0 + 1$. If a_1 is the greatest member of A_1 , I plays the ω^{m_0} th point of B_1 from the right. Now II must reply in A_2 , so I again wins in 2 more moves.

If $m_0 = 1$, we have to show that I can win in 3 moves. As this is not the exceptional case, A_2 or B_2 has a least, and by the above remark they both have a least. Hence $A_2 = \omega^{n_0}$ and $B_2 = \omega^{q_0}$. I plays the ω th point b_1 of B_1 from the right. In this case, all members of A have successors, so whatever point of A II plays, I can win in 2 more moves.

The basis cases for 3.5(i) are the following: $A_1 = \omega^{m_0}$, $B_1 = \omega^{p_0} \cdot (\omega^*)^{p_1} \dots$; $A_1 = (\omega^*)^{m_0}$, $B_1 = (\omega^*)^{p_0} \cdot \omega^{p_1} \dots$; $A_1 = \omega^{m_0}(\omega^*)^{m_1}$, $B_1 = \omega^{m_0}(\omega^*)^{p_1} \dots$; $A_1 = (\omega^*)^{m_0}\omega^{m_1}$, $B_1 = (\omega^*)^{m_0}\omega^{p_1} \dots$.

In the first of these, A_1 has a least and B_1 does not, and the same applies to A and B , so $l(A, B) = l(A_1, B_1) = 1$. In the third and fourth cases, since we are working on the copies of ω^{m_0} or $(\omega^*)^{m_0}$ an auxiliary game is used, and I can force the play to lie on the left (or else II deviates to the right and I wins quickly).

The second case is one of the exceptional ones, so has already been covered.

The basis case for 3.4(ii) is $s = u = 2$, $A_1 = M_0^{m_0}M_1^{m_1}$, $B_1 = M_0^{p_0}M_1^{p_1}$, $m_0 < p_0$. We write $A' = M_1^{m_1}$, $B' = M_0^{p_0-m_0}M_1^{p_1}$. Here A has a greatest element or least, and B' has neither, so I wins $G(A', B')$ in 2 moves. He plays according to this winning strategy in $G(A, B)$ playing the least or greatest element of the appropriate copy of $M_0^{m_0}$ in $A_1 \cup B_1$, and wins in at most $2m_0 + 2$ moves (considering separately the case in which II plays in $A_2 \cup B_2$ on one of the first two moves).

The basis case for 3.5(ii) is handled similarly, which is $A_1 = M_0^{m_0}M_1^{m_1}$, $B_1 = M_0^{p_0}M_1^{p_1}M_2^{p_2} \dots$, $m_0 \neq p_0$.

The basis cases for 3.4(iii) and 3.5(iii) are $s = u \geq 3$, $A_1 = M_0^{m_0} \dots M_{s-1}^{m_{s-1}}$, $B_1 = M_0^{p_0} \dots M_{u-1}^{p_{u-1}}$, $m_0 < p_0$. The same argument applies, on $M_0^{m_0}$ blocks, taking into account dealing with a possible play by II on the right. ■

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