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ON THE SPECTRUM OF THE LAPLACIAN ON A STRIP WITH VARIOUS BOUNDARY CONDITIONS

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Abstract

In this paper, the spectrum of the Laplace operator on a strip with constant width subject to four different boundary conditions is investigated. In all the four situations, we prove that its spectrum starts from the first eigenvalue of the one-dimensional Laplacian considered along the width of the strip. Unlike the other cases, we demonstrate that in the case of Robin boundary conditions, the negative part of the spectrum is not necessarily empty and establish sufficient conditions for this to happen.

1. Introduction

It is well known that the operator $-\Delta$ densely defined on the space $L^2(\mathbb{R}^n)$ is self-adjoint and its essential spectrum is equal to $[0, \infty)$, which is absolutely continuous. However, in the case of a strip, the bottom of its essential spectrum depends on the boundary conditions. Our main interest in the present paper is to describe the location of the spectrum of the Laplacian

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on a straight strip subject to various boundary conditions. The precise description of the problem studied here is as follows:

Let $a > 0$ and $S = \mathbb{R} \times I$, where $I = (0, a)$, with Neumann (N), Dirichlet (D), Dirichlet-Neumann (DN) or Robin (R) boundary conditions. Consider the following spectral problem:

$$\begin{cases} -\Delta u = \lambda u & \text{in } S, \\ B_l u = 0 & \text{on } \partial S, \end{cases} \tag{1}$$

where B_l is one of the boundary operators, and

$$\mathrm{Dom}(-\Delta) = \{u \in W_2^2(S) : B_l u = 0\},$$

where $W_2^2(S)$ denotes the standard Sobolev space $H^2(S)$, that is, the space of square integrable weak derivatives up to the second order (see, e.g., [1] for details). The operators on the transverse section I, $-\Delta_l^I$, are the usual Laplacian on $L^2(I)$ with Dirichlet boundary conditions if $l = D$, the Neumann conditions if $l = N$, the Dirichlet at 0 and the Neumann one at a if $l = DN$ or the Robin conditions if $l = R$. Robin conditions can be considered as a generalization or a linear combination of the Dirichlet and Neumann boundary conditions (see (3)).

2. Dirichlet, Neumann and Dirichlet-Neumann Boundary Conditions

Let $l \in \{D, N, DN\}$. Assume that (1) has a non-trivial solution of the form

$$u(x, y) = X(x)Y(y), \quad X \neq 0, Y \neq 0.$$

Then one has

$$-\frac{X''(x)}{X(x)} = \frac{Y''(y)}{Y(y)} + \lambda = C \text{ in } S$$

for some suitable separation constant C. This sort of separation of variables gives rise to two independent one-dimensional spectral problems, that is, the longitudinal and the transverse one. The spectrum of the longitudinal Laplacian $(-\Delta^{\mathbb{R}})$ on $L^2(\mathbb{R})$ is the positive-real semi-axis, i.e.,

$$\sigma(-\Delta^{\mathbb{R}}) = \sigma_{ess}(-\Delta^{\mathbb{R}}) = [0, \infty).$$

The eigenvalues of the transverse Laplacian $(-\Delta_l^I)$ on $L^2(I)$ are given by

$$\lambda_n^D := \left(\frac{\pi}{a}\right)^2 n^2, \quad \lambda_n^N := \left(\frac{\pi}{a}\right)^2 (n-1)^2, \quad \lambda_n^{DN} := \left(\frac{\pi}{2a}\right)^2 (2n-1)^2, \qquad (2)$$

where $n = 1, 2, \ldots.$ The corresponding normalized eigenfunctions $\{f_n\}_{n=1}^{\infty}$ are given as follows:

$$f_n^l(y) := \sqrt{\frac{2}{a}} \sin \sqrt{\lambda_n^l}\, y \ \text{ for } l \in \{D, DN\},$$

$$f_n^N(y) := \begin{cases} \sqrt{\dfrac{1}{a}} & \text{if } n = 1, \\ \sqrt{\dfrac{2}{a}} \cos \sqrt{\lambda_n^N}\, y & \text{if } n \geq 2. \end{cases}$$

Since the eigenfunctions f_n^l form a complete orthonormal set in $L^2([0, a])$ by Fourier analysis, there are no other eigenvalues apart from those listed in (2) (see, e.g., [2] for more details). The description of the spectral properties of (1) for $l \in \{D, N, DN\}$ can also be found in [3, 4, 7] and the references therein.

Theorem 2.1 [2, Theorem 4.1.5]**.** *The essential spectrum of a self-adjoint operator H on a Hilbert space is empty if and only if there is a complete set of eigenfunctions* $\{f_n\}_{n=1}^{\infty}$ *of H such that the corresponding eigenvalues* λ_n *converge in absolute values to* ∞ *as* $n \to \infty$.

Thus, by the above theorem and (2), $\sigma_{ess}(-\Delta_l^I) = \emptyset$.

Theorem 2.2 [Weyl criterion]**.** *Let A be a self-adjoint operator on a Hilbert space $\mathcal{H}$. A point $\lambda \in \mathbb{R}$ belongs to $\sigma_{ess}(A)$ if and only if there exists a sequence $\{f_n\}_{n\in\mathbb{N}} \subset D(A)$ such that for all $n \in \mathbb{N}, \| f_n \| = 1,$ f_n converges weakly to* 0 *and*

$$\| Af_n - \lambda f_n \| \to 0 \ \text{ as } \ n \to \infty.$$

Moreover, $\{f_n\}$ can be chosen as an orthonormal.

Theorem 2.3 (See, e.g., [8, Section 10.1, Theorem 1])**.** *Let $\{f_n\}_{n\in\mathbb{N}}$ be a bounded sequence in a Hilbert space $\mathcal{H}$. If $\langle f_n, g\rangle_{\mathcal{H}} \to 0$ as $n \to \infty$ for g in a dense subspace of $\mathcal{H}$, then $f_n \to 0$ weakly in $\mathcal{H}$.*

Theorem 2.4 (cf. [6, Theorem 4.1])**.** $\sigma(-\Delta_l^S) = \sigma_{ess}(-\Delta_l^S) = [\lambda_1^l, \infty)$.

Proof. Let $\mathcal{E}_I^l[u]$ and $\mathcal{E}_S^l[u]$ denote the quadratic forms of the Laplacian on *I* and *S*, respectively, subject to the boundary conditions *l*. Since $\sigma(-\Delta_l^I)$ starts by $\lambda_1^l,$ for all $u \in \text{Dom}(-\Delta_l^I),$ we have

$$\mathcal{E}_I^l[u] = \int_0^a | u_y |^2 dy \geq \lambda_1^l \| u \|_{L^2(I)}^2.$$

Now,

$$\begin{aligned}
\mathcal{E}_S^l[u] &= \int_S | u_x |^2 dxdy + \int_S | u_y |^2 dxdy \\
&\geq \int_S | u_y |^2 dxdy \\
&= \int_{\mathbb{R}} dx \left(\int_0^a | u_y |^2 dy \right) \\
&\geq \lambda_1^l \| u \|_{L^2(S)}^2.
\end{aligned}$$

This implies that $\sigma(-\Delta_l^S) \subseteq [\lambda_1^l, \infty)$.

On the other hand, pick $\varphi \in C_0^\infty(\mathbb{R})$ with $\operatorname{supp}\varphi = [-1, 1]$ such that $\|\varphi\|_{L^2(\mathbb{R})} = 1$. Let $\varphi_n(x) := n^{-\frac{1}{2}}\varphi\left(\frac{x}{n}\right)$ so that $\|\varphi_n\|_{L^2(\mathbb{R})} = 1$.

Take $\forall\lambda \geq \lambda_1^l$ and consider a sequence $\{u_n\}_{n=1}^\infty \subset \operatorname{Dom}(-\Delta_l^S)$ given by

$$u_n(x, y) := \varphi_n(x)e^{i\sqrt{\lambda-\lambda_1^l}x} f_1^l(y).$$

Then

$$\|u_n\|_{L^2(S)} = 1.$$

Since $-(f_1^l)''(y) = \lambda_1^l f_1^l(y)$ and

$$\Delta u_n(x, y) = \varphi_n''(x)e^{i\sqrt{\lambda-\lambda_1^l}x} f_1^l(y)$$
$$+ 2i\sqrt{\lambda - \lambda_1^l}\varphi_n'(x)e^{i\sqrt{\lambda-\lambda_1^l}x} f_1^l(y) - \lambda\varphi_n(x)e^{i\sqrt{\lambda-\lambda_1^l}x} f_1^l(y),$$

$$-\Delta u_n(x, y) - \lambda u_n(x, y) = -\varphi_n''(x, y)e^{i\sqrt{\lambda-\lambda_1^l}x} f_1^l(y)$$
$$- 2i\sqrt{\lambda - \lambda_1^l}\varphi_n'(x)e^{i\sqrt{\lambda-\lambda_1^l}x} f_1^l(y)$$

implying that

$$\|(-\Delta_l^S - \lambda)u_n\| \leq \|\varphi_n''\|\|f_1^l\| + 2\sqrt{\lambda - \lambda_1^l}\|\varphi_n'\|\|f_1^l\|$$
$$= \frac{1}{n^2}\|\varphi''\| + 2\sqrt{\lambda - \lambda_1^l}\frac{1}{n}\|\varphi'\| \to 0 \text{ as } n \to \infty.$$

Now, it remains to show that $u_n \to 0$, $n \to \infty$ weakly in $L^2(S)$. For any $N \in \mathbb{N}$, let

$$w_N(x, y) := \begin{cases} w(x, y), & \text{if } |w(x, y)| \leq N \\ 0, & \text{if } |w(x, y)| > N \end{cases}$$

with $w \in W_2^1(S)$. Then $w_N \in L^1(S) \cap L^2(S)$ and $\| w - w_N \|_{L^2(S)} \to 0$ as $N \to \infty$. Also,

$$| \langle u_n, w_N \rangle_{L^2(S)} | \leq \| u_n \|_{L^\infty(S)} \| w_N \|_{L^1(S)} \leq \frac{\text{const}}{\sqrt{n}} \| w_N \|_{L^1(S)} \to 0$$

as $n \to \infty$.

Since $\| u_n \|_{L(S)} = 1$, u_n converges weakly to 0 in $L^2(S)$ by Theorem 2.3. Thus, Theorem 2.2 implies that $\lambda \in \sigma_{ess}(-\Delta_l^S)$ and

$$[\lambda_1^l, \infty) \subseteq \sigma_{ess}(-\Delta_l^S) = \sigma(-\Delta_l^S). \qquad \square$$

3. Robin Boundary Conditions

In this section, we discuss in detail the spectrum of the Laplacian on a straight strip with Robin boundary conditions. We shall see that the negative part of its spectrum is not necessarily empty as opposed to the cases of Neumann, Dirichlet and Dirichlet-Neumann boundary conditions. Theorem 2.4 and its proof remain true for the case of Robin boundary conditions but the quadratic form of the Laplacian involves boundary terms.

Let $S_0 := [0, 1] \times [0, a]$. Consider the following eigenvalue problem:

$$\begin{cases} -\Delta u = \lambda u \text{ in } S_0, \\ u_x(0, y) = u_x(1, y) = 0, \\ u_y(x, 0) + \alpha u(x, 0) = 0, \\ u_y(x, a) + \beta u(x, a) = 0 \end{cases} \tag{3}$$

for $\alpha, \beta, \lambda \in \mathbb{R}$. A related type of problem has been studied in [5], that is,

$$\begin{cases} -\Delta u = \lambda u \text{ in } S, \\ u_y(x, 0) - \alpha(x) u(x, 0) = 0, \\ u_y(x, a) + \alpha(x) u(x, a) = 0, \end{cases} \tag{4}$$

where $\alpha(x)$ is positive for all $x \in \mathbb{R}$. Under the hypothesis that $\alpha(x)$ tends to a constant as $|x| \to \infty$, the essential spectrum of (4) was determined and a sufficient condition for the existence of the discrete spectrum was established. However, in this section, we only study the location of the bottom of the spectrum when one chooses the boundary conditions in (3). We establish sufficient conditions for the bottom of the spectrum to lie on negative part of the real line. Assume that a solution of (3) has the form

$$u(x, y) = v(x)w(y).$$

Then (3) reduces to two one-dimensional problems, namely:

$$\begin{cases} -v''(x) = (\lambda - \tau)v(x), \quad 0 < x < 1, \\ v'(0) = 0, \\ v'(1) = 0 \end{cases} \tag{5}$$

and

$$\begin{cases} -w''(y) = \tau w(y), \quad 0 < y < a, \\ w'(0) + \alpha w(0) = 0, \\ w'(a) + \beta w(a) = 0, \end{cases} \tag{6}$$

where $\tau \in \mathbb{R}$ is a separation constant.

The solution of (5) is given by

$$v(x) = \cos m\pi x, \quad \lambda = \tau + \pi^2 m^2, \quad m = 0, 1, 2, \ldots. \tag{7}$$

To solve (6), we consider the following cases:

(i) For $\tau = 0$, the solution of the ordinary differential equation in (6) is of the form $w(y) = Ay + B$ for some constants A and B. The first boundary condition implies that

$$w(y) = -\alpha y + 1 \quad (\text{take } B = 1).$$

This implies that $\tau = 0$ is in the spectrum if and only if the following condition holds true:

$$(1 + \beta a)\alpha = \beta. \tag{8}$$

Hence, when $\tau = 0$, the solutions of (3) are given by

$$u(x, y) = \cos m\pi x(1 - \alpha y), \quad \lambda = \pi^2 m^2, \quad m = 0, 1, 2, \ldots. \tag{9}$$

(ii) $\tau > 0$ gives the following general solution:

$$w(y) = A\cos\sqrt{\tau}y + B\sin\sqrt{\tau}y.$$

The boundary conditions in (6) yield

$$w(y) = \cos\sqrt{\tau}y - \frac{\alpha}{\sqrt{\tau}}\sin\sqrt{\tau}y \quad (\text{take } A = 1) \tag{10}$$

and

$$\tan\sqrt{\tau}a = \frac{(\beta - \alpha)\sqrt{\tau}}{\tau + \alpha\beta}. \tag{11}$$

Thus, we get a sequence $\tau_n = \theta_n^2$, $n = 1, 2, \ldots$ satisfying:

(a) as $\alpha, \beta \to 0$, $\theta_n \to \frac{n\pi}{a}$,

(b) as $\alpha, \beta \to \pm\infty$, $\theta_n \to \frac{n\pi}{a}$,

(c) as $\alpha \to 0$, $\beta \to \pm\infty$, $\theta_n \to \frac{(2n+1)\pi}{2a}$,

(d) as $\beta \to 0$, $\alpha \to \pm\infty$, $\theta_n \to \frac{(2n+1)\pi}{2a}$.

The related eigenfunctions are

$$w_n(y) = \left\{\cos\theta_n y - \frac{\alpha}{\theta_n}\sin\theta_n y\right\}_{n=1, 2, \ldots}. \tag{12}$$

Hence, the solutions of (3) become

$$u(x, y) = \cos m\pi x\left(\cos\theta_n y - \frac{\alpha}{\theta_n}\sin\theta_n y\right),$$

$$\lambda = \theta_n^2 + \pi^2 m^2, \quad m = 0, 1, 2, \ldots, \quad n = 1, 2, \ldots. \tag{13}$$

If $\alpha = \beta$, then one gets $\tau = \left(\frac{n\pi}{a}\right)^2$, $n = 0, 1, 2, \ldots$. Thus, (13) becomes

$$u(x, y) = \cos m\pi x\left(\cos\frac{n\pi}{a}y - \frac{\alpha a}{n\pi}\sin\frac{n\pi}{a}y\right),$$

$$\lambda = \left(\frac{n\pi}{a}\right)^2 + \pi^2 m^2, \quad m = 0, 1, 2, \ldots, \quad n = 0, 1, 2, \ldots. \tag{14}$$

See [9] for more details.

Also, a special case of (11): $\tau = -\alpha\beta$, then $\cos\sqrt{\tau}a = 0$, i.e., $\tau = \left(\frac{(2n+1)\pi}{2a}\right)^2$, $n = 0, 1, 2, \ldots$. So, this case occurs if and only if $\alpha\beta = -\left(\frac{(2n+1)\pi}{2a}\right)^2$ for some n. Hence, (13) becomes

$$u(x, y) = \cos m\pi x\left(\cos\frac{(2n+1)\pi}{2a}y - \frac{2\alpha a}{(2n+1)\pi}\sin\frac{(2n+1)\pi}{2a}y\right),$$

$$\lambda = \left(\frac{(2n+1)\pi}{2a}\right)^2 + \pi^2 m^2, \quad m = 0, 1, \ldots, \quad n = 0, 1, \ldots. \tag{15}$$

(iii) Let $\tau_1 < \tau_2$ be the smallest eigenvalues of (6). For some values of α and β, τ_1 or τ_2 might be negative. Suppose that $\tau = -\sigma^2$ $(\sigma > 0)$ is a negative eigenvalue. Then (10) and (11), respectively, become

$$w(y) = \cosh(\sigma y) - \frac{\alpha}{\sigma}\sinh(\sigma y), \tag{16}$$

$$\tanh(\sigma a) = \frac{(\beta - \alpha)\sigma}{-\sigma^2 + \alpha\beta}. \tag{17}$$

To investigate when this happens, note that (8) divides the (α, β)-plane into three connected components and the number of negative eigenvalues in each of them is the same since eigenvalues are continuous with respect to α

and β. Consider the line $\beta = -\alpha$, it transects all the three regions (see Figure 1). By applying the shift $y \mapsto y - b$, $b = \frac{a}{2}$, (6) becomes

$$\begin{cases} -w''(y) = \tau w(y), & -b < y < b, \\ w'(-b) + \alpha w(-b) = 0, \\ w'(b) - \alpha w(b) = 0. \end{cases} \tag{18}$$

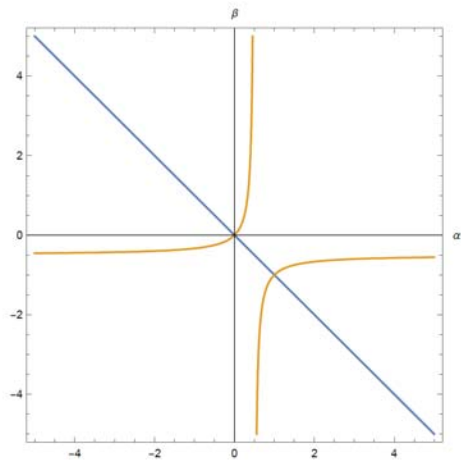

Figure 1. $(1 + \beta a)\alpha = \beta$, $\beta = -\alpha$.

So, if $w(y)$ is an eigenfunction, then the function $Q(y) = w(-y)$ is also an eigenfunction with the same eigenvalue. Thus, we can consider separately the eigenfunctions that are even functions and those that are odd functions, described, respectively, by

$$\begin{cases} -w''(y) = \tau w(y), & 0 < y < b, \\ w'(0) = 0, \\ w'(b) - \alpha w(b) = 0 \end{cases} \tag{19}$$

and

$$\begin{cases} -w''(y) = \tau w(y), & 0 < y < b, \\ w(0) = 0, \\ w'(b) - \alpha w(b) = 0. \end{cases} \tag{20}$$

Considering $w(y) = \cosh(\sigma y)$ and $w(y) = \sinh(\sigma y)$, the boundary conditions in (19) and (20), respectively, yield

$$\alpha = \sigma \tanh(\sigma b) \tag{21}$$

and

$$\alpha = \sigma \coth(\sigma b). \tag{22}$$

For $\sigma > 0$, both the functions $\sigma \tanh(\sigma b)$ and $\sigma \coth(\sigma b)$ are monotone increasing with minima equal to 0 and $\frac{1}{b}$, respectively, at $\sigma = 0$. Thus, (21) has one solution if and only if $\alpha > 0$, and (22) has one solution if and only if $\alpha b > 1$. Hence, if $\alpha a > 0$, then there is one negative eigenvalue with even eigenfunction and if $\alpha a > 2$, then another negative eigenvalue comes from odd eigenfunction.

In general, we have the following situations: If

(a) $\alpha + \alpha\beta a - \beta < 0$ and $\alpha < \frac{1}{a}\left(\beta > -\frac{1}{a}\right)$, then $\tau_1 > 0$,

(b) $\alpha + \alpha\beta a - \beta = 0$ and $\alpha < \frac{1}{a}\left(\beta > -\frac{1}{a}\right)$, then $\tau_1 = 0$ and $\tau_2 > 0$,

(c) $\alpha + \alpha\beta a - \beta > 0$, then $\tau_1 < 0$ and $\tau_2 > 0$,

(d) $\alpha + \alpha\beta a - \beta = 0$ and $\alpha > \frac{1}{a}\left(\beta < -\frac{1}{a}\right)$, then $\tau_1 < 0$ and $\tau_2 = 0$,

(e) $\alpha + \alpha\beta a - \beta < 0$ and $\alpha > \frac{1}{a}\left(\beta < -\frac{1}{a}\right)$, then $\tau_1 < 0$ and $\tau_2 < 0$.

Cases (c)-(e) produce the following solutions of (3):

$$u(x, y) = \cos m\pi x\left(\cosh \sigma_n y - \frac{\alpha}{\sigma_n}\sinh \sigma_n y\right),$$

$$\lambda = -\sigma_n^2 + \pi^2 m^2, \quad m = 0, 1, 2, \ldots, \tag{23}$$

where $n = 1$ in Cases (c) and (d) while $n = 1, 2$ in Case (e).

Thus, by Theorem 2.4 and (iii) above, we have the following:

Proposition 3.1. *Let* α *and* β *satisfy any of the conditions* (c)-(e) *above. Then* $(-\infty, 0) \cap \sigma(-\Delta_R^S) \neq \emptyset.$

Remark 3.2. When $\alpha = \beta(\neq 0)$, it is shown in [9] that $\lambda_1 = -\alpha^2$ and that $\lambda = 0$ is not an eigenvalue.

When $\alpha = 0$, we have the Neumann conditions at 0 and the Robin conditions at a. By (8), $\tau = 0$ if and only if $\beta = 0$. If $\beta > 0$, then $\tau_1, \tau_2 > 0$ by condition (a) above. If $\beta < 0$, then $\tau_1 < 0$ and $\tau_2 > 0$ by condition (c) above.

When $\beta = 0$, we have the Robin condition at 0 and the Neumann conditions at a. By (8), $\tau = 0$ if and only if $\alpha = 0$. If $\alpha > 0$, then $\tau_1 < 0$ and $\tau_2 > 0$ by condition (c) above. If $\alpha < 0$, then $\tau_1 > 0$ and $\tau_2 > 0$ by condition (a) above.

Let $\alpha \to \pm\infty$. Then we have the Dirichlet conditions at 0 and the Robin conditions at a. Again, (8) implies that $\tau = 0$ if and only if $1 + \beta a = 0$. Equations (16) and (17), respectively, become

$$w(y) = \sinh(\sigma y) \tag{24}$$

and

$$\tanh(\sigma a) = -\frac{\sigma}{\beta}. \tag{25}$$

Now, (25) implies that

$$\beta = -\sigma \coth(\sigma a) \tag{26}$$

(cf. (22)). Hence, $\tau_1 > 0$ if $1 + \beta a > 0$ and $\tau_1 < 0,\ \tau_2 > 0$ if $1 + \beta a < 0$.

Let $\beta \to \pm\infty$. Then we have Robin conditions at 0 and Dirichlet conditions at a. By (8), $\tau = 0$ if and only if $\alpha a - 1 = 0$, and (17) becomes

$$\alpha = \sigma \coth(\sigma a) \tag{27}$$

(cf. (22)). Thus, $\tau_1 < 0$, $\tau_2 > 0$ if $\alpha a - 1 > 0$ and $\tau_1 > 0$ if $\alpha a - 1 < 0$.

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