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Building on Students' Prior Mathematical Thinking: Exploring Students' Reasoning Interpretation of Preconceptions in Learning Mathematics

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Abstract: *The present study explored 285 11th-grade students' preconceptions, misconceptions, and errors in solving mathematics tasks by graphical method. A descriptive-explorative study design was adopted. Cluster sampling was used to select students from sampled secondary schools in eastern and central Uganda. Students' paper and pen solution sketches together with the task-based interviews were used to identify students' preconceptions, misconceptions, and errors in linear programming (LP). Students' responses were analyzed thematically and interpreted as students' learning gaps. The results indicated that students lacked proficiency in relating basic algebraic concepts and procedures to the mathematical language with which LP is conveyed. Generally, most students could not adequately use their previous knowledge and connect it to the learning and solving of LP tasks. Besides applying wrong mathematical algebraic concepts, students had difficulties interpreting and writing correct models (inequalities) from LP word problems. Misconceptions and errors were common and peculiar to and in individual student's solution sketches especially in applying the concepts of equations and inequalities to graphically solve and optimize LP tasks. Students held extremely weak concept images of graphing equations, and inequalities and their linkage to optimizing feasible regions. This research provides insight into the learning of mathematics word problems (LP) and recommends that mathematics educators should effectively apply students' preconceptions, misconceptions, and errors as opportunities for enhancing students' LP conceptual changes. For mathematical proficiency, suitable learning approaches, methods, and strategies should be adapted to address specific individual student's flawed conceptual and procedural knowledge and understanding. These approaches may guide educators in helping students to construct the connections between the old and the new knowledge.*

INTRODUCTION

Research in the 21st-century education system on learning science and mathematics has taken a differential trend. Sometimes and most often pre-existing knowledge may aid or hinder acquisition of students' subsequent conceptual knowledge and understanding. The learning of mathematics is still more challenging for both students and for some teachers as it requires both parties to have a deeper and broader mathematical conceptual understanding of basic algebraic concepts and principles (NCTM, 2014). There exists a relationship between students' prior knowledge and understanding, and the learning of new concepts. Indeed, how teachers use students' prior knowledge and understanding to improve their pedagogy is what may define effective learning. This scenario is called teachers' pedagogical content knowledge. Research shows that assessing students' prior scientific and mathematical knowledge and understanding allows educators to plan, focus, adopt, and adapt new learning strategies (Dong et al., 2020; Otero & Nathan, 2008). To students, prior mathematical knowledge helps them to construct connections between the old and new knowledge. Thus, students' understanding of mathematical concepts can be improved by teachers reviewing prior knowledge and understanding before introducing new concepts. The implication is that students may understand better when educators review their prior knowledge and understanding, and effectively link it to subsequent learning. Students' understanding of new concepts may explicitly influence knowledge acquisition and capacity to apply higher-order cognitive problem-solving skills.

Educators can identify students' proficiency/learning gaps by reviewing their prior mathematical knowledge and understanding (Celik & Guzel, 2017). These may include pointing out previous and subsequent topics to be covered, providing lesson roadmaps, inviting reflective problem-solving tasks, and application of active learning activities like concept maps or case studies. These strategies can be implemented in small groups or for individual students taking into account their learning gaps and differences. Research (e.g., Celik & Guzel, 2017; Rach & Ufer, 2020) shows that using students' prior knowledge, understanding and experiences, may help in generating practical examples through scaffolding learning for making connections to increase knowledge acquisition and retention. Thus, educators can use students' prior knowledge and understanding to identify the learning gaps (preconceptions, misconceptions, and errors), and topical learning difficulties, justify why students are struggling and consequently correct the flawed concepts.

According to the cognitive load theory, the information learned must be held in the working memory until it has been processed sufficiently to pass into the long-term memory to acquire highly complex knowledge and skills (Kalyuga & Singh, 2016; Paas & Ayres, 2014). Therefore, the teachers' pedagogical content knowledge is potentially key in identifying students' prior conceptions and effectively using preconceptions to improve pedagogy. According to Shulman (1986), pedagogical content knowledge (PCK) refers to "the most useful forms of representation of those ideas, the most powerful analogies, illustrations, examples, explanations, and demonstrations in a word, the ways of representing and formulating the subject that make it comprehensible to others" (p.9). Shulman argued that the teachers' PCK dimension is unique and combines pedagogical knowledge (PK) and content knowledge (CK). Thus, application of prior knowledge schemata is significant in restructuring the learning process (Galili & Goldberg, 2001).

The teachers' PCK relates to teachers' effectiveness in enhancing students' achievement and proficiency in a typical classroom learning environment. Generally, PCK involves the teachers' interpretations and transformations of the subject-matter knowledge (SMK) in the context of facilitating effective learning. Indeed, what differentiates mathematics teachers from other scientists is how the knowledge of teaching is organized and used to foster students' learning. Thus, the learning practices should be used effectively in supporting specific content (NCTM, 2014). This may involve teachers' competencies in delivering the conceptual approach, relational understanding and adaptive reasoning of the subject matter (Kathirveloo & Marzita, 2014). This knowledge component is what Hill et al. (2008) referred to as mathematical knowledge for teaching (MKT) and the mathematical quality of instruction (MQI); the unique knowledge that intersects with the specific subject teacher characteristics to produce effective and meaningful instruction. According to Hill, "teachers with weak MKT would have teaching characterized by few affordances and many deficits". Hill further noted elements for MQI as those that involve dealing with students' mathematical errors, responding to students appropriately, connecting classroom practice to mathematics in real-life, mathematical language and richness of the mathematics (p. 437).

Some empirical studies conducted in different settings and contexts (e.g., Baumert et al., 2010; Kleickmann et al., 2015; Cankoy, 2010; Halim et al., 2013; Jong, 2018) have demonstrated the significance of the above knowledge dimensions (PCK, MKT and MQI) in enhancing students' understanding of science and mathematics. In understanding the learning of mathematics and LP in particular, some studies (e.g., Kenney et al., 2020; Shikuku, 2017) show that the topic of LP is challenging and that students have limited conceptual understanding of equations and inequalities. These factors impede students' understanding of basic concepts (e.g., gradient, equations and inequalities) and its application in graphing (Dolores-Flores et al., 2021). The causes of students' learning challenges in LP are enormous and mainly stem from preconceptions of equations and inequalities. Students, on one hand, fail to understand the relationship between equations, inequalities, and LP while teachers on the other hand have not adequately applied suitable learning approaches to address the causes and sources of students' flawed conceptions (misconceptions and errors). The annual UNEB reports on students' performance support the above claim (UNEB., 2020, 2019, 2018, 2016). The above learning challenges (and other related factors) have limited students' conceptualization of LP and related concepts.

Yet, LP is applied in vast areas of science, technology, engineering, and mathematics (e.g., Dhal, 2016; Parlesak et al., 2016; Romeijn et al., 2006; van Dooren, 2018; Von Gadow & Walker, 1982; Wilen & Fadel, 2012). For instance, LP concepts are necessary for finding solutions to everyday non-routine mathematics problems and in making relevant decisions in management. Indeed, LP is a subset of operations research that aims to optimize scarce resources as a result of opportunity cost in different sectors of any country's economy (e.g., business, engineering, manufacturing, medical, etc.). In learning LP, many students come to class with previously learnt and developed concepts, ideas, principles and particular flawed ways of thinking and reasoning. Thus, the purpose of this study was to use LP mathematics tasks to explore students' preconceptions, misconceptions and errors in LP. It is expected that this research may provide insight to mathematics educators in using students' preconceptions as a springboard for devising suitable pedagogies. This can be achieved by providing varied and meaningful instruction to enhance students' conceptual and procedural knowledge and understanding. This is because students may persistently retain flawed prior knowledge even after undergoing an effective instruction.

The Conceptual Framework

This research is situated on the PCK conceptual framework based on Shulman (1986). Shulman conceptualized that "pedagogical content knowledge also includes an understanding of what makes the learning of specific topics easy or difficult: the conceptions and preconceptions that students of different ages and backgrounds bring with them to the learning of those most frequently taught topics and lessons" (p. 9). According to Shulman, effective learning strategies involve teachers' integration of students' preconceptions and misconceptions held previously and how these preconceptions relate to subsequent learning. In supporting students' mathematical thinking and understanding, Taşdan & Çelik (2016) developed a framework for examining mathematics teachers' PKC. The framework is related to Shulman's conceptualization of PCK, and is important in enhancing teachers' PCK (e.g., the use of graphics, manipulatives) with the main objective of understanding students' mathematical thinking.

Indeed, the above theoretical framework aligns with the five strands of mathematical proficiency. Kilpatrick, Swafford and Findell (2001) proposed a multidimensional five interwoven and interdependent strands of mathematical proficiency teachers should target during the classroom instruction. These strands are conceptual understanding, procedural fluency, strategic competence, adaptive reasoning, and productive disposition (NCTM, 2014). Kilpatrick, Swafford and Findell have argued "that proficiency should enable them to cope with the mathematical challenges of daily life and enable them to continue their study of mathematics in high school and beyond" (p. 116). The above five interrelated strands are inevitable for learning mathematics in the sense that they foster, support and promote students' identification and acquisition of conceptual knowledge, procedural knowledge, problem-solving skills, abilities, and beliefs. These skills are all supported by the cognitive load theory. However, educators should ask themselves the effective ways of motivating learners to represent and connect prior knowledge and understanding and effectively use it deeply and broadly during problem solving. According to NCTM (2014), students' effective learning "depends fundamentally on what happens inside the classroom as teachers and learners interact over the curriculum" (p. 8). To support the learning process, and enhance pedagogy, Stein et al. (1996) resonated that students' proficiency and competency is determined by mathematical tasks they are given. Tasks at the lower cognitive stage (memorization level), for example, must be different from those at the highest cognitive level (doing mathematics).

The Study Context

In this paper, students were engaged by prospective teachers with classroom scenarios grounded on the notion that learning and teaching are inseparable, and that students' previous knowledge and understanding are inevitable for subsequent learning. We drew research experiences from a cohort of 285 students to highlight and identify students' preconceptions, misconceptions, and errors in learning LP. The tasks provided to students exhibited model-eliciting activities (Hamilton et al., 2008; Lesh et al., 2000). In a typical classroom scenario, these tasks are perceived as those having suitable principles to adequately measure students' knowledge and understanding in LP. In particular, the tasks measured students' abilities in model formulation from word problem statements, problem-solving, and critical thinking. Thus, the tasks were engaging and challenging. Students' paper and pen responses provided feedback on their preconceptions with the main objective of enhancing pedagogy through variation of instructional practices.

The tasks were suitable for engaging the 11th grade (locally known as senior four class) students within the Ugandan lower secondary school curriculum (see Appendix 1). It was predicted that when prospective teachers provide suitable tasks to students based on their cognitive level, their critical thinking, conceptual understanding, procedural fluency and problem-solving abilities may be enhanced. This may thus address students' learning gaps. This can be achieved by teachers varying their classroom learning approaches, methods and strategies. Thus, teachers are engaged to help students work collaboratively in small groups or individually. The purpose of engagement is to help students make sense of, and apply previous mathematical ideas, principles, rules to guide them reason mathematically when learning new concepts.

This research mainly focused on the learning of mathematics and linear programming (LP) in particular. Linear programming is one of the topics taught to the 11th grade Ugandan lower secondary school students (NCDC, 2008, 2018). At this level, the graphical solution of LP problems and in two dimensions (x, y) is mostly emphasized (other methods e.g., simplex method is outside the scope of this study). Students' prior conceptual knowledge and understanding of equations and inequalities is a prerequisite for learning LP word problems. The Ugandan lower secondary school syllabus content (8th grade to the 11th grade) outlines the application of equations and inequalities in finding optimal solutions to LP problems. The 8th grade is locally referred to as senior one class while the 11th grade is senior four class, the terminal class where students sit for national examinations (UNEB). Moreover, the Ugandan lower secondary school curriculum materials are more procedural than conceptual. This partly aligns with the conceptual framework of this study. An example of a LP problem at this level adapted for this study is shown in question 1 below. This particular LP task was adapted from Ugandan mathematics curriculum materials (textbooks, teachers' reference books and past paper national examinations).

Question 1.

A Geography club in a certain school wishes to go for the field work excursion to a national park. The club hired a **mini-bus** and a **bus** to take students. Each trip for the bus had to cost Shs.**500,000** and that of a mini-bus, Shs.**300,000**. Due to Covid-19 pandemic, the bus transported **36** students and the mini–bus, **9** students. The maximum number of students allowed to go for the excursion was **216**. The number of trips the bus made did not exceed those made by the mini-bus. The club had mobilized Shs.**3,000,000** for the transportation of students.

(i) Write down five inequalities representing the above information.

(ii) Plot a suitable graph for the inequalities in (i) shading out the unwanted regions.

(iii) How many journeys should the bus and mini-bus make so as to minimize transport?

We recognize that question 1 above was challenging to students. Students spend at least 30 minutes thinking about this problem. Most of them wrote and represented different models (both wrong and some correct) on separate Cartesian coordinates, which could not yield optimal solutions. While others plotted wrong models. The teachers' responses and engagement was aimed at supporting, fostering and improving students' understanding. This paper aims to demonstrate the effect of students' preconceptions in understanding and learning LP. This task (and others) were applied to help teachers identify their PCK necessary for engaging learners for deeper and broader understanding. We drew our conclusions from teachers' responses for effective pedagogy aimed at enhancing the learning of LP. It is expected that this study will improve teachers' pedagogical strategies for learning and solving LP (and related) tasks. The above LP problem in question 1 can be symbolically written in the form:

min./max. $x_1 \pm x_2$

$s.t\ ax_1 \pm bx_2 \leq\geq c$

$dx_1 \pm ex_2 \leq\geq f$

$gx_1 \pm hx_2 \leq\geq i$

$x_1 \geq 0$

$x_2 \geq 0$

To adequately solve the above LP problem, the non-negative constraints $(x_1 \geq 0)$ and $x_2 \geq 0$ and the set of points satisfying the main constraints ($ax_1 \pm bx_2 \leq\geq c, dx_1 \pm ex_2 \leq\geq f$ and $gx_1 \pm hx_2 \leq\geq i$) are plotted on the same coordinate axes, the feasible region is identified at the region of intersection of the lines representing inequalities (after shading unwanted regions). The corner points of the bounded half-plane constraint set are substituted into (min./max. $x_1 \pm x_2$), called the objective function, and used for finding optimal solutions.

Methodology

The Research Subjects and their Classroom Experiences

This study involved 285 students (126 male and 159 female). The subjects were engaged in solving LP tasks (by graphical method). It was an activity-based three months mathematics interventional research study designed to investigate how students' prior knowledge and understanding could aid subsequent mathematical learning, reasoning, and problem-solving. Whereas LP may appear to be an independent topic, equations and inequalities form prerequisite conceptual knowledge for understanding this topic. This means students need to develop a better conceptual understanding of equations and inequalities before learning LP. All sampled students participated in the study. All the subjects were the 11th grade students who were preparing to write national examinations for the academic year 2020/2021. Students had varied mathematical conceptual background. To correlate paper and pen responses with students' learning experiences, focus group discussions and interviews were conducted with students and also with their respective teachers. Students were interviewed as a group and where possible individually after completing a unit on LP which lasted for four weeks with each week covering 4 hours of class periods (three times a week, 80 minutes per lesson). The sub-topics discussed in class during the learning of LP included formation of equations and inequalities from mathematics word problems (appendix 1), plotting equations and inequalities on the same coordinate axes and optimization of the feasible region.

The Research Design

This study adopted an exploratory-descriptive study design. Exploratory-descriptive designs are used to collect data in natural settings to explain phenomena from the perspective of the persons being studied (Creswell, 2014). This design was useful in summarizing and understanding student's previous mathematical thinking. It was appropriate for this study in describing and exploring students' preconceptions and their understanding of LP. This was done with the view of restructuring students' thinking by constructing meaning from equations and inequalities to adequately help students graphically solve LP and related tasks. The goal was to help students to have a complete understanding of LP and to have a critical and creative understanding of solving non-routine LP word problems by graphical method. The research subjects were all the 11th grade students from eastern and central Uganda. It was predicted that the sampled students had had prior knowledge and understanding of LP and related concepts at the time of data collection.

FINDINGS

The aim of this study was to investigate student's prior mathematical thinking and explore how their reasoning and interpretation of preconceptions can enhance the learning of mathematics. Data was coded and analyzed and discussed qualitatively following clearly defined steps (Miles et al. 2014). The mathematical content of question 1 above involving the solution of LP tasks (by graphical method) for optimizing the objective function subject to the constraints is central as outlined in the 11th grade Ugandan mathematics syllabus, developed by the national curriculum development centre (NCDC, 2018).

The notion of graphical solutions of LP problems encompasses several concepts and mathematical principles (Appendix 1) which are sequentially learned from 8th grade to 11th grade. For students' conceptual understanding, the stated LP prior concepts (Appendix 1) should be reviewed by mathematics teachers before introducing and/or learning LP. The Ugandan mathematics textbook and related reference materials have been designed procedurally. In particular, this research focused on students' prerequisite knowledge in solving LP tasks by graphical method and how teachers applied students' preconceptions knowledge component to enhance the learning of LP.

Analysis of LP Task(s)

Students were interviewed by the principal researcher assisted by four research assistants. The focus group interview (Appendix 2) with selected mathematics teachers consisted of a set of tasks arranged in a sequence, each aiding the learning and solving of LP tasks by graphical method. All audio recordings of interviews were transcribed verbatim. In each case, the selected students were also asked related tasks to write models from the given mathematics word problem, represent models (inequalities) on the same coordinate axes, shed out the unwanted regions and finally optimizing the objective region. To obtain supportive information, students were also asked to solve a series of additional tasks related to the learning of equations and inequalities and their relationship with LP. Students were finally asked to describe their conceptual understanding of LP concepts as indicated in question 1.

Students' failed to comprehend mathematics word problems, and also wrote incorrect inequalities (please see vignettes in Appendix 3). . This was the first step and perhaps the greatest hindrance to attaining optimal solutions. Most students could not mathematize LP word tasks by transforming a mathematics word problem into correct models (inequalities). There were conceptual flaws and misinterpretations across the group of students. Moreover, the preconceptions, misconceptions and errors were relatively consistent amongst the majority of individual students. This suggests that most students shared a common conceptual challenge generally attributed to the linkage between equations, inequalities, and the learning of LP. This was heavily attributed to students limited or poor command of English language vocabulary and its relationship to mathematics (exceed, at least, at most, greater than, less than, minimize, maximize, etc.). Yet, on the beliefs about access and equity in learning mathematics, the NCTM (2014) asserts that "Students who are not fluent in the English language are less able to learn mathematics and, therefore, must be in a separate track for English language learners" (p. 63). Thus, teachers are argued to promote students' meaningful learning by formulating effective approaches, methods, and techniques. This research may help educators to identify a community of resources to understand how students use contexts, culture, conditions, and language to support mathematical learning.

The coding of students' interaction was performed deductively (Kooloos et al. 2020) while the coding of interpretation and decisions was inductive. This helped to explore the teacher's interpretation of their students' thinking, and the decisions they made in solving LP tasks. Over 90 percent of students were unable to interpret, link mathematical symbols to mathematical meaning. Consequently, some students failed to write correct equations and inequalities, used wrong equations to obtain coordinates, and plotted incorrect graphs by separating instead of plotting them simultaneously on the same coordinate axes. This meant that they could neither identify the correct feasible region nor get corresponding coordinates for optimization. Some students could not decide on whether or not to shade wanted or unwanted regions. Therefore, they were unable to solve and optimize the given LP problem.

Table 1. Rubric for Classifying Strategies used in Learning Linear Programming

Correct use of the Strategy	Partial use of the Strategy	Limited or no use of the Strategy
5 points	3 points	1 point
Evidence of students' insightful thinking in LP question interpretation and problem exploration.	Limited clarity on question interpretation and writing some wrong models.	There is completely limited students' understanding of LP word statements from which models are written.
Learner's procedures for solving LP problems are all clear and focused.	Learners apply some correct strategies which yield some partial correct answers.	The procedures have no relationship with the questions asked
Appropriate strategies are applied and demonstrate a students' understanding and thinking of basic concepts.	The learner starts the problem appropriately but fails to apply some of the basic algebraic concepts for solving LP problems, and loses focus.	The learner does not completely understand the basic LP concepts to be used to fully explore the problem at hand linking to the strategies.
The learner portrays a clear understanding and provides extensions or generalizations of procedures and possible strategies to the solution of the given problem.	Learners may recognize procedures, and/or patterns but fail to correctly identify relationships hence applying concepts incorrectly.	The learner portrays incorrect or no understanding of concepts and steps, hence providing no extensions or generalizations of procedures and possible strategies to the solution of the given problem.

From Table 1 above, 156 students mixed up the concepts by shading unwanted regions on separate graphs. At least 80 percent of students plotted graphs of inequalities instead of equations while others interchanged the axes (x and y respectively) and could not adequately interpret the feasible region. Students' inability to identify the feasible (accessible) region, identify integral values, and use the objective function to optimize the stated LP problem was the major hindrance. Thirty-five students completely failed to identify the feasible region even after writing the other correct procedures. Fifty-six students had correctly represented equations (transformed from inequalities) from word problem statements but failed to plot them and/or identify the points of intersection, x-intercepts and y-intercepts from graphs. Thus, these particular students wrote wrong coordinates and hence wrong optimal solutions. All these challenges partly stemmed from students' English language deficiency. Below is an extract of focus group interview transcripts conducted with some students.

Students' Argumentation on the Learning of Linear Programming

During the interview process, students expressed their conceptual knowledge and understanding, elaborating the prior knowledge they held. The teachers together with the research team engaged students to understand their underlying concepts. In these transcriptions, the selected students struggled to apply prior knowledge in writing correct models to represent word statements. The students wrote wrong or incomplete models since they could not mathematize statements related to the learning of LP. This provided an opportunity to explore students' understanding and effectively use it to build on their thinking. In some instances, however, the teachers confirmed and where necessary, corrected wrong models to help students plot correct graphs. The following is an excerpt of selected students' preconceptions.

Interviewer: Read for me question 1.

Student: She reads fluently.

Interviewer: interpret and summarize what the question is all about?

Student: Sir, some words are confusing.

Interviewer: Which words are confusing?

Student: Number of trips the bus make did not exceed those made by the mini-bus.

Interviewer: Okay! Which symbol correctly represents this statement?

Student: Sir, it is $<$

Interviewer: Why not $>, \leq or \geq$?

Student: Because they should not exceed.

Interviewer: Okay! Now write all the five inequalities.

Student: $x \geq 0;\ y > 0;\ 500{,}000x + 300{,}000y = 3{,}000{,}000; 36x + 9y \geq 216\ and\ x < y$

Interviewer: Are all the above inequalities, correct?

Student: Yes sir.

Interviewer: May you please tell me the meaning of using the inequality symbols $<, \leq, >, \geq$?

Student: Sir, they confuse me. They mean more than and less than when you read and interpret tasks.

Interviewer: How will you get coordinates from this inequality $500{,}000x + 300{,}000y = 3{,}000{,}000$?

Student: Sir, you just remove all zeros.

Interviewer: Okay, write the final inequality

Student: $5x + 3y = 3$

Interviewer: Using the above inequality, get your coordinates to be plotted.

Student:

x	0	0
y	1	0.6

Interviewer: How do you plot the pair of coordinates (0, 0.6)?

Student: Sir, you get the scale.

Interviewer: How do you plot coordinate for all the five lines?

Student: You plot each line separately on the graph and shade unwanted regions

Interviewer: now, how do you get the feasible region?

Student: Okay, after plotting separate lines, you now combine and plot all lines on the same graph, shade outside and leave the middle part as the feasible region.

Interviewer: Is the feasible part always in the middle after shading the outer side of each equation?

Student: Of course, sir. Our teacher told us to do so.

Interviewer: Now, how do you minimize transport?

Student: You get some coordinates from the middle of the feasible region and substitute them in the equation.

Interviewer: Which equation?

Student: Sir, it is not given in this question.

From the above verbal transcription, we can infer some characteristics of students' common conceptions and misconceptions arising from the interview transcripts (see vignettes in Appendix 3). Students were asked to write the procedure for optimizing a LP word problem (see question 1) and explain the process of optimization. It was evident that most students had flaws in answering LP problem 1 and other related LP tasks. However, the students' responses to the above problem provided evidence for exploring the best instructional practices for enhancing the learning of LP and related concepts. Thus, teachers must be aware of students' prior preconceptions, misconceptions and errors in terms of their beliefs and incomplete understandings that may directly or indirectly conflict with subsequent learning. Teachers should also create an enabling platform and circumstances where students' LP learning gaps can be externalized, expressed and discussed explicitly.

DISCUSSIONS

This study applied a descriptive-explorative study design to explore 285 11th-grade students' preconceptions in solving LP tasks by graphical method. From the above findings, it is evident that students faced learning challenges in LP. This perhaps explains the practicability of LP in our daily lives outside the classroom environment. It reveals students' conceptual and learning gaps in applying mathematics in solving societal problems. This research acknowledges the fact that LP is one of the challenging mathematics word problems topics. Thus, educators should teach LP sequentially for students' conceptualization. From the above research findings, and to effectively teach this topic, teachers need to teach from simple to complex and from concrete to abstract to build on the pre-existing mathematical ideas and principles. Students may be guided to develop and understand challenging and more complex mathematical concepts. There is need for prospective teachers to include students' preconceptions during lesson preparation and instruction to minimize misconceptions and consequently mathematical errors. Lastly, the competence-based curriculum (CBC) recently introduced in Ugandan lower secondary school curriculum emphasizes that teachers should innovatively by providing students with suitable learning materials so that they use them to observe and make practical experiments. The use of learning materials develops and improves students' generic skills. Inclusion of students' preconceptions, misconceptions and errors exposes their potential, weaknesses and learning gaps which can be corrected during instruction (Kooloos et al., 2022). According to NCTM (2014), scientific explanations backed with clarity and, with possible, examples lead to better conceptual understanding. Teachers should, therefore, allot adequate time with individual commitment to engage, explore, explain, elaborate and evaluate their learning objectives. This is a restructured and blended mode of instruction to counter students' prior misconceptions and errors. To achieve this, learning should be designed in such a way that mathematics topics are taught in a sequence and that the content should be taught deeply and broadly than covering many topics in a superficial manner.

Evidence from this study indicates that students failed to link prior conceptual knowledge and understanding to the learning of LP concepts which consequently affected subsequent learning. It can further be noted that some students had not adapted the learning environment at the time of data collection. Some students who had joined the sampled schools during their 10th and 11th grades could not fully solve LP and related tasks. Thus, to such students, the knowledge they had acquired cannot be represented as a well-defined intermediate state. This research adds to other empirical findings (e.g., Connell, 2015; Kooloos et al., 2022; Nelson, 1992), and suggests that teachers should develop a strong restructuring between pre-instruction, instruction and evaluation of students' learning outcomes. The teachers' conceptions of the relevant mathematical concepts may play a significant role in interpreting and correcting students' flawed concepts. Prospective teachers should inculcate these learning principles by enhancing their mathematical knowledge for teaching and mathematical quality of instruction. This may help to overcome students' challenges arising from preconceptions, misconceptions and errors. The research conducted by Kooloos et al.

(2022) on teachers' orientations toward using student mathematical thinking as a resource during whole-class discussion shows that teachers might be supported in their novice attempts at whole-class discourse by explicitly discussing students' conceptions and the learning gaps.

Comprehension of LP word problems accounted for students' misconceptions and errors. Fifty-four percent of students made these errors (see vignettes in Appendix 3). Comprehension errors varied based on students' previous academic abilities, background and preconceptions. At least 70% of students had difficulty understanding LP questions. Due to students' inability to interpret mathematical word problems, wrong inequalities were written. Consequently, erroneous models were utilized to represent and solve the LP task (question 1), and this resulted in inaccurate solution sets. This was mainly attributed to students' limited understanding and proficiency in English language and its relationship to the interpretation and application of basic mathematical principles. The interviews conducted demonstrate a link between students' English language development, mathematics background, and the learning environment, all of which hampered students' knowledge acquisition.

Students failed to translate LP mathematical word statements from English to their individual "mother tongue" languages and vice versa, as well as the connection to symbolic representations. This contributed to most of the misconceptions and errors in LP. The findings are in agreement with Pongsakdi et al. (2020), and, Makonye and Fakude (2016) findings since students are constantly confronted with linguistic challenges, relational challenges, and interactions between linguistic and symbolic representations. Generally, students had limited mathematical vocabulary as evidenced by their incorrect use of mathematical symbolism ($<$, $>$, $\leq$, and $\geq$) representing statements conveyed in the question. Other related phrases wrongly interpreted included "at least, at most, greater than, "less than," "minimize," "maximize, "feasible" etc." In contrast to their peers in urban settings, these errors were seen more often in students' paper and pen works from those studying from rural secondary schools.

At least half (50%) of students were unable to link mathematical symbols, variables, constraints, mathematical operations, algebraic expressions to mathematical language, and the solution of inequalities and LP in particular (see students' vignettes in Appendix 3). Students' inability to consistently use symbols ($=$, $<$, $>$, $\leq$ and $\geq$) to represent numbers rather than objects and expressing them in mathematical sense contributed to most misconceptions and errors. Many students who were interviewed could not distinguish the mathematical interpretation of the symbols $\leq$ from $\geq$, relating to "at least" and "at most," respectively. Similarly, the traditional sense and usage of $<$and $>$ was perplexing, and this led to wrong interpretations and solution sets. Students repeatedly failed to make clear distinction between the symbols ($<$, $>$), and ($\leq$, $\geq$), and their contextual applications. Students' understanding of the graphical solution of LP problems was hampered by this mystery. Many of them could not write correct equations from inequalities or could not correctly plot equations on the x-y plane.

Over 80% of students were unable to appropriately represent the specified inequalities (albeit some inequalities were incorrect) on the Cartesian plane. In addition, inequalities that were incorrectly written could not yield optimal solutions (see vignettes in Appendix 3). The majority of students (99% of low achievers) failed to use the graphical approach appropriately in finding the feasible region. To some students, it was debatable whether or not to draw graphs equations or inequalities. The graphical solution to LP problems became more complicated in the case where two or more inequalities were to be plotted on the same coordinate axes. Students instead plotted graphs on different coordinate axes without taking into account their intersection in this scenario. Thus, they could not identify the feasible region. This was due to the repetitious procedures that some students found difficult to comprehend. Students' inability to shade the correct side of the graph to obtain the feasible area resulted in a mix of graphical and arithmetic difficulties. Furthermore, some students were unable to transform inequalities to equations and vice versa prior to obtaining the appropriate coordinates for plotting on coordinate axes. Students plotted the graphs of inequalities instead of equations, illustrating a mismatch between inequalities and equations. To illustrate this concept, students plotted the graph of the inequality $y \geq x\text{-}a$, instead of $y = x\text{-}a$. This means the Cartesian coordinates could then be obtained using the equation $y = x\text{-}a$ before sketching the graph of $y = x\text{-}a$.

Procedural difficulties were the main cause of skill manipulation. The failure by some students to understand LP word problems, as well as their inability to reason analytically and insightfully, had an impact on the entire process. At least 60% of students struggled with the option of whether or not to draw dotted and solid lines to express inequalities with mathematical symbols < or >, and ≤ or ≥ respectively. For question 1, 72% of students failed to correctly shade unwanted regions and isolate the feasible region (see vignettes in Appendix 3). Between numeric and symbolic representations, there were many misunderstandings and overgeneralizations. It is possible that this occurred owing to a lack of prior knowledge of equations and inequalities and their relationship to the learning of LP. When it came to identifying the feasible area, which is defined as an intersection of at least two constraints with relational symbols of <, >, ≤ or ≥, there were numerous observable inconsistencies, all of which led to wrong optimal solutions.

Even after defining the feasible region correctly, some students (42%) were unable to use the objective function to find the numerical optimized solution to question 1. As a result, the stated numerical answers were wrong. It is possible that students found the reversal of this procedure more baffling or intimidating. Research by Botty et al., (2015), Kenney et al., (2020) and Tsamir and Almog (2001) supports the research findings. Furthermore, some students who correctly identified the feasible region were unable to write correct integral and/or "corner coordinates" from the feasible region which consequently led to writing wrong models (equations and inequalities). Yet, the correct objective function was to be written for students to optimize the LP problem 1. Correct coordinates were to be extracted and appropriately used in writing models. It was, however, observed that erroneous graphs were plotted in some cases resulting in inaccurate feasible regions.

Students also made careless mistakes (Appendix 3). As mentioned earlier, this could have stemmed from poor command of English language. This mostly affected low and average achievers who were unable to write correct inequalities from word problem statements. Although high achievers also made these types of errors, they were not common. Approximately 18% of high achievers were unable to correctly use the scale and align the axes. Consequently, some students failed to write the necessary corner coordinates (integral coordinates) for optimization or derive suitable coordinates from equations for plotting graphs. Inconsistencies in shading wanted regions instead of unwanted regions, interchanging the axes (x and y, respectively), failure to distinguish points of intersection, x-intercepts, and y-intercepts from graphs, and failure to obtain coordinates from equations and/or inequalities and their inability to map them were major challenges. Students' failed to interpret the graph and accurately answer the tasks as portrayed in the question. During face-to-face interviews, some students couldn't interpret and connect their cognitive schema to the given LP problems. When representing dotted and solid lines for the graphs of $y \geq x-4$ and $y>-3x$, some students couldn't distinguish the difference between the processes and relational inequality symbols. Specifically, some students found it difficult to graphically represent equations of the form $y=a$ and $x=a$, hence failing to plot the graph of $y>-3x$. To the majority of students, it was difficult to relate $y=a$ and $x=a$ in representing a horizontal line or a vertical line.

Conclusion

The purpose of this study was to use LP mathematical tasks to explore 285 grade 11 students' preconceptions, misconceptions and errors in learning the topic of LP. The findings show that students generally held weak and flawed algebraic concepts (equations and inequalities). Students failed to link prior conceptual knowledge and understanding of equations and inequalities to the learning of LP concepts. Comprehension of LP word problems accounted for most of the students' misconceptions and errors. Students failed to translate LP mathematical word problem statements. They could not link mathematical symbols, variables, constraints, and mathematical operations, algebraic expressions to mathematical language, and the solution of inequalities and LP (see students' vignettes in Appendix 3). Students struggled to distinguish equations from inequalities and their relationship to the learning of LP. Some students failed to represent inequalities graphically. Thus, they could not identify the feasible region. To some students, even after defining the feasible region correctly, they failed to use the objective function to find the numerical solution. These (and related factors) at large limited students' understanding of and finding suitable solutions to sampled LP problems. The findings provide practical insight in using students' preconceptions to aid subsequent learning. This was an exploratory study with inherent limitations. Future researchers in different settings and contexts may apply quantitative and mixed methods approached to compare and contrast the stated research findings. In conclusion, adopting and consistently applying students' preconceptions in learning mathematics can be a key ingredient and factor in identifying students' flawed learning gaps which is aimed at achieving students' mathematical proficiency. It is likely that using suitable LP and related mathematics tasks in the typical classroom and teacher training education contexts can be employed to:

1. Identify and explore mathematical, didactical, and pedagogical issues. There is need for student engagement that may trigger students' reflection of preconceptions about the learning of mathematics and LP in particular.
2. Apply LP tasks as opportunities for preparing and addressing students' learning challenges.
3. Apply LP tasks to develop teachers' pedagogical content knowledge and their awareness in addressing students' flawed concepts.

4. Use students' preconceptions, misconceptions and errors as opportunities for addressing and developing teachers' continuous professional development programs.
5. Applying mathematics in real-life scenarios beyond the compulsory level at the 11th grade.

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Appendix 1

Eleventh Grade Students' Content Understanding and Key Mathematical Ideas for Learning the Graphical Method of Solving Linear Programming Problems

Content (with Related Examples) for Solving LP Tasks	Key Mathematical Ideas for Learning LP in Secondary Schools
1. Representation of inequalities on a number line and writing down the solution set. e.g., Using a number line, find the integral values of x which satisfy the sets: $\{3x > 2x + 5\} \cap \{3x < 32 - x\}$ 2. Solving linear inequalities algebraically (non-graphical). e.g., Solve: $\frac{x-2}{4}$ - $\frac{x-2}{3} < 1$	▪ Correct use of a number line and related symbols. ▪ Knowledge of equations, i.e., the gradient of a straight line, negative gradient, positive gradient, finding the equation of the straight line in the form $y = \pm mx \pm c$, $y = \pm mx, y = \pm m, x = \pm m$, etc. ▪ Solving linear equations correctly including the use of lowest common multiples, simplifying fractions, etc. ▪ Correct use of set notations e.g., $\cup, \cap, \emptyset, \in$, etc. ▪ The distinction between solving equations and inequalities. ▪ Correct use of symbols in solving linear equations.
3. Solving quadratic inequalities algebraically (non-graphical). e.g., Solve: $3x^2 + 7x - 20 < 0$	▪ Identification of correct factors (critical points). ▪ Distinguishing critical values from solutions to the quadratic inequality. ▪ Review of the methods of solving quadratic equations. Showing students differences between the following graphs $y = \pm ax^2, y = \pm x^2$, $y = \pm ax^2 \pm bx, y = \pm ax^2 \pm bx \pm c$, $y = (x \pm p)^2$, $y = (x \pm p)^2 + k$, $y = \pm a(x \pm p)(x \pm p), = \pm(x \pm p)(x \pm p)$ etc. substitute = with symbols like $>, <, \geq$ and $\leq$. Students' knowledge of $b^2 - 4ac = 0$, $b^2 - 4ac \leq 0$ and $b^2 - 4ac \geq 0$ ▪ Consistent use of mathematical symbols ▪ The distinction between inequalities with equations. ▪ Review of representation of equations and inequalities on the coordinate axes.
4. Solving linear inequalities graphically	▪ Obtaining correct coordinates from the given equations. ▪ Distinguishing dotted from solid lines based on mathematical symbols $<, >, \leq, \geq$.

e.g., Show by shading unwanted regions the region satisfying the inequalities: $x + y \leq 3$, $y > x - 4$, and $y \geq -3x$	▪ Identification of the correct feasible region. ▪ Correct use of scales on the coordinate axes. ▪ Decision on whether to shade wanted/unwanted regions. ▪ Identification of correct coordinates for plotting and testing the objective function to find the feasible region. ▪ Plotting graphs of equations instead of inequalities.
5. Solving simultaneous inequalities graphically. e.g., On the same coordinate axes, draw the curve $y= 4x^2$ and the line y = 1. Show by shading unwanted regions, the region represented by: $y > 1$ and $y < 4x^2$. Hence, state the integral coordinates of the points which lie in the region $\{y > 1 \cap y < 4x^2\}$	▪ The distinction between graphs of curves and straight lines. ▪ Correct use of mathematical symbolism when representing the coordinate axes. ▪ Failure to obtain coordinates from equations. ▪ Obtaining correct coordinates from the feasible region. ▪ Obtaining correct integral values and convex points.
6. The graphical solution of a LP problem. e.g., A school has organized a geography study tour for 90 students. Two types of vehicles are needed; Taxis and Costa buses. The maximum capacity of the taxi is 15 passengers while that of the Costa bus is 30 passengers. The number of taxis will be greater than the number of Costa buses. The number of taxis will be less than five. The cost of hiring a taxi is Shs.60,000 while that of the Costa bus is Shs. 100,000. There is only Shs. 600,000 available. (a) If x represents the number of taxis and y the number of Costa Buses, write inequalities for the given information.	▪ Students' comprehension of mathematical word problems, translating and writing correct inequalities from mathematics word statements. ▪ Students' ability to link mathematical symbols, variables, constraints, operations, algebraic expressions into mathematical language. ▪ Correct use of scales when representing equations on the coordinate axes. ▪ Writing the correct models and the objective function for optimization. ▪ Students' ability in solving equations simultaneously by graphical means. ▪ Review of basic mathematical vocabulary (at least, at most, greater than, less than, minimize, maximize, etc. ▪ Emphasize finding the correct feasible region. ▪ Identification of correct coordinates for optimization. ▪ Decision on shading unwanted regions and leaving out wanted regions.

(b) Represent the inequalities on the graph paper by shading the unwanted regions. (c) Find from your graph the number of taxis and Costa Buses which are full that must be ordered so that all the students are transported? (d) Find the minimum and maximum cost of transporting 90 students?	▪ Adequate knowledge of the graphical solution of linear and quadratic equations simultaneously. ▪ Plotting graphs of inequalities, not equations. ▪ Correct labeling and use of axes (x and y respectively). ▪ Identification of correct points of intersection, x-intercepts, and y-intercepts from the graph. ▪ Ability to obtain coordinates from equations, and correctly plotting them on the same coordinate axes. ▪ Identifying the correct feasible region, obtaining integral values, and optimizing the coordinates. ▪ Ability to write correct equations, inequalities from the given feasible region. ▪ Interpretation of optimization terms (maximum or minimum), this leads to correct or incorrect substitutions and numerical values. ▪ Ability to use symbols to represent numbers, not objects.

Appendix 2

Dear participant,

You are requested to answer the following questions related to the effective learning of linear programming. This interview is likely to take approximately 45 minutes. It is expected that the answers arising from our interaction are aimed at improving the learning of linear programming in secondary schools.

1. What is linear programming within the context of secondary school mathematics?
2. What specific students' prior knowledge is necessary for the learning of linear programming?
3. Differentiate between equations, inequalities, and linear programming.
4. What are the challenges of teaching linear programming as compared to other topics?
5. What are the causes of students' challenges in learning linear programming?
6. When can the learning of linear programming concepts be introduced?
7. What techniques and procedures do you apply when teaching linear programming?
8. How do you make students understand these techniques and procedures?
9. Do these techniques and procedures become more complex as students' progress?
10. What learning materials are used to help students grasp linear programming concepts?
11. What steps do you consider inevitable for students to understand linear programming?
12. What procedures are followed when introducing (teaching) linear programming concepts?
13. Which aspects of linear programming are most problematic for students to comprehend?
14. How do you help your students to overcome the challenges of learning linear programming?
15. How do you identify students with learning challenges in linear programming?
16. What else are you doing to help students understand linear programming concepts?
17. In your own opinion, what should be done to improve the learning of linear programming?

Looking at question 6 in Appendix 1:

a. What concepts are necessary for illustrating and solving this question?
b. How many standard marks can be allocated for this question?
c. How can the concepts be broken down to help students answer the question effectively?

d. Briefly explain how you will allocate marks for this particular question.
e. Solve this question the way you would expect students to solve it.
f. What concepts are easily grasped by students in this question?
g. What concepts limit subsequent learning of this question?
h. Some teachers say this question is usually hard for most students. What do you say?
i. Some teachers say this topic is challenging to introduce and teach. What is your view?
j. Do students elude LP questions during national examinations? If yes, why?

Thank you for your feedback.

Appendix 3

Students' Vignettes on LP Model Formulation and Graphing Abilities