Countably categorical coloured linear orders

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In this paper, we give a classification of (finite or countable) $\aleph_0$-categorical coloured linear orders, generalizing Rosenstein's characterization of $\aleph_0$-categorical linear orderings. We show that they can all be built from coloured singletons by concatenation and $\mathbb{Q}_n$-combinations (for $n \geq 1$). We give a method using coding trees to describe all structures in our list.

1 Introduction

A coloured linear ordering is a triple $(A, <, F)$ where $(A, <)$ is a linear order and F is a mapping from A onto a set C which we think of as a set of colours. We shall sometimes simply write A instead of $(A, <, F)$. The *orbit* of an n-tuple $(a_1, a_2, \ldots, a_n)$ of members of A is the set of all its images under the action of the automorphism group of A (where automorphisms are required to preserve colour as well as order), and an *n-orbit* of A is the orbit of some n-tuple. We also say that A is *almost n-tuply transitive* if it has a finite number of n-orbits. In particular, if $n = 2$, we say that A is *almost doubly transitive*, and A is *homogeneous* if any isomorphism between finite substructures can be extended to an automorphism.

An n-type is a realizable set of formulae having free variables among $x_1, x_2, \ldots, x_n$ for some $n \geq 1$. We note that an n-type corresponds to an n-orbit, and we shall therefore use the two terms interchangeably. Let $S_n(T)$ denote the set of complete n-types of a theory T. Engeler, Ryll-Nardzewski and Svenonius independently proved that a theory T is $\aleph_0$-categorical if and only if every $S_n(T)$ is finite.

Using the Engeler-Ryll-Nardzewski-Svenonius Theorem, Rosenstein [4] characterized which linear orders have $\aleph_0$-categorical theories as follows. Let Δ be the smallest class of order-types containing 1, and closed under concatenation of finitely many elements, and under 'shuffle' (which is the same as our '$\mathbb{Q}_n$-combination' – see below).

Theorem 1.1 (Rosenstein) *A complete theory T of linear orderings is $\aleph_0$-categorical if and only if it has a model whose order-type is in Δ.*

It is our goal in this paper to generalize Theorem 1.1 to coloured linear orderings. As in [1] and [2] we are able to describe the structures in terms of 'coding trees'. This is not essential in the finitely coloured case, but still provides a clear way of describing all the examples. In [2] the case where the colour set is infinite is considered, and there coding trees are indispensible, but for us it is necessarily finite, because if the set of colours is infinite then so is the set of 1-types, contrary to $\aleph_0$-categoricity.

The *n-coloured version* $\mathbb{Q}_n$ of the rationals is the set $\mathbb{Q}$ of rational numbers, together with a colouring function $F : \mathbb{Q} \longrightarrow C$ where $|C| = n \geq 1$, such that between any two distinct points, there are points of all possible colours. (See [1] for instance.) We write $\mathbb{Q}_n(Z_0, Z_1, \ldots, Z_{n-1})$ for the countable coloured linear ordering resulting from replacing all points coloured $c_i \in C$ by Z_i for $i = 0, 1, \ldots, n-1$ where $Z_0, Z_1, \ldots, Z_{n-1}$ are given coloured linear orders and $C = \{c_0, c_1, \ldots, c_{n-1}\}$. We shall refer to $\mathbb{Q}_n(Z_0, Z_1, \ldots, Z_{n-1})$ as a *$\mathbb{Q}_n$-combination*. If $Z_0, Z_1, \ldots, Z_{n-1}$ are also pairwise disjoint, then their *concatenation* is the coloured linear order $\bigcup_{i \in n} Z_i$ ordered by $x < y$ if $x, y \in Z_i$ and $x < y$ for some i or $x \in Z_i$ and $y \in Z_j$ where $i < j$, coloured by the union of the colourings on the individual Z_i.

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2 The main proof

We first introduce two equivalence relations $\sim_1$ and $\sim_2$ on any coloured linear order A (the first of which applies to any linear order, not necessarily coloured) to carry out the 'condensation' process needed in our classification.

Equivalence relation $\sim_1$: Let $\sim_1$ be given by $x \sim_1 y$ if $x \leq y$ and $[x, y]$ is finite or $y \leq x$ and $[y, x]$ is finite. We note that the $\sim_1$-classes are clearly convex, and so $A/\sim_1$ receives an induced linear order from that on A. Although this can be carried out for any linear order, in our case we want it to be coloured, and the way that this is done is explained below.

Lemma 2.1 *For any linear ordering A, $\sim_1$ is an equivalence relation.*

Proof. Reflexivity and symmetry are immediate.

For transitivity, suppose $x \sim_1 y$ and $y \sim_1 z$, and without loss of generality, suppose that $x \leq z$. Then if $z \leq y$, $[x, z] \subseteq [x, y]$, if $x \leq y \leq z$, $[x, z] \subseteq [x, y] \cup [y, z]$, and if $y \leq x$, then $[x, z] \subseteq [y, z]$. In each case, $[x, z]$ finite follows from $[x, y]$ and $[y, z]$ finite. So $x \sim_1 z$. □

Lemma 2.2 *If A is an $\aleph_0$-categorical coloured linear ordering, then all the $\sim_1$-classes are finite, of bounded size.*

Proof. From the definition of $\sim_1$, it follows that its classes are finite or have order-type ω, ω^* or $\omega^* + \omega$, where ω^* is the reverse ordering of ω. We note that ω, ω^* and $\omega^* + \omega$ cannot arise, since then A would have infinitely many 2-types contrary to $\aleph_0$-categoricity, so the classes are finite. The fact that their sizes are bounded also follows from the finiteness of the number of 2-types. □

This result enables us to define a colouring on $A/\sim_1$ in a natural way. Since A is $\aleph_0$-categorical, there are finitely many orbits of singletons (1-types) and from this it follows that there are finitely many orbits of $\sim_1$-classes. We take these as the colours on $A/\sim_1$, and then each $\sim_1$-class is coloured by its isomorphism type.

Lemma 2.3 *If L is an almost doubly transitive* (*coloured*) *linear ordering, then L is almost n-tuply transitive for each $n \geq 2$.*

Proof. Let the number of 2-orbits of pairs $\langle x, y\rangle$ in L with $x < y$ be q. We shall show that the number of orbits of strictly increasing n-tuples in L is at most q^{n-1}, which clearly implies that the number of n-orbits is finite. Suppose that $a_1 < a_2 < \cdots < a_n$ and $b_1 < b_2 < \cdots < b_n$ are such that for each $j \leq n-1$, $\langle a_j, a_{j+1}\rangle$ and $\langle b_j, b_{j+1}\rangle$ are in the same 2-orbit. Let h_j be an automorphism (colour-preserving) of L such that $h_j(a_j) = b_j$ and $h_j(a_{j+1}) = b_{j+1}$. Since $h_j(a_{j+1}) = h_{j+1}(a_{j+1}) = b_{j+1}$, we can define an automorphism h of L by patching h_1 on $(-\infty, a_2]$, h_2 on $(a_2, a_3]$, . . ., and h_{n-1} on $(a_{n-1}, +\infty)$. It follows that $\langle a_1, a_2, \ldots, a_n\rangle$ and $\langle b_1, b_2, \ldots, b_n\rangle$ are in the same n-orbit. Hence the n-orbit of $\langle a_1, a_2, \ldots, a_n\rangle$ is fully determined by the sequence of 2-orbits of $\langle a_1, a_2\rangle, \langle a_2, a_3\rangle, \ldots, \langle a_{n-1}, a_n\rangle$. Hence L has at most q^{n-1} orbits of increasing n-tuples. □

Lemma 2.4 *If A is an $\aleph_0$-categorical coloured linear ordering, then under the induced ordering and the colouring by orbits of $\sim_1$-classes, $A/\sim_1$ is also $\aleph_0$-categorical.*

Proof. By definition, the 1-types of $A/\sim_1$ are precisely the orbits of $\sim_1$-classes of A, and there are finitely many of these. One deduces from the fact that A has finitely many 2-types that the number of 2-types of pairs $\langle (a)_{\sim_1}, (b)_{\sim_1}\rangle$ in $A/\sim_1$ such that $(a)_{\sim_1} < (b)_{\sim_1}$ is also finite. By Lemma 2.3 $A/\sim_1$ is n-tuply transitive for each $n \geq 1$, and hence is $\aleph_0$-categorical. □

Lemma 2.5 *For any linear ordering A, $A/\sim_1$ is either a singleton or is densely ordered by the induced relation.*

Proof. Let $(x)_{\sim_1} < (y)_{\sim_1}$ in $A/\sim_1$. Since $x \not\sim_1 y$, $[x, y]$ is infinite, but as $(x)_{\sim_1} \cup (y)_{\sim_1}$ is finite by assumption, there must be some $z \in [x, y] - ((x)_{\sim_1} \cup (y)_{\sim_1})$. This z is $\sim_1$-inequivalent to both x and y, so $(x)_{\sim_1} < (z)_{\sim_1} < (y)_{\sim_1}$. □

Equivalence relation $\sim_2$: Let C' be a subset of the colour set C of A. Let $x \sim_2 y$ if either $x = y$ or all points of $[x, y]$ if $x \leq y$ or of $[y, x]$ if $y \leq x$ are coloured by members of C'. Thus as C' varies we get different equivalence relations. Note that all non-singleton equivalence classes must consist of points all of whose colours lie in C'.

Lemma 2.6 *For any C-coloured linear ordering A, and subset C' of C, $\sim_2$ is an equivalence relation.*

P r o o f. Clearly $\sim_2$ is reflexive and symmetric. For transitivity, suppose that $x \sim_2 y \sim_2 z$. If $x = y$ or $y = z$ or $x = z$, then $x \sim_2 z$ is immediate, so we suppose all are distinct. Hence all points between x and y and between y and z have colours in C'. Hence the same applies to all points between x and z, so $x \sim_2 z$. □

Once again the equivalence classes are convex and so $\sim_2$ induces a linear ordering on $A/\sim_2$. (A special case of $\sim_2$ is the one where $x \sim_2 y$ if $x = y$ or $x \leq y$ and $[x, y]$ is monochromatically c or $y \leq x$ and $[y, x]$ is monochromatically c, which is obtained by taking $C' = \{c\}$ of size 1.) The quotient $A/\sim_2$ becomes a coloured linear ordering with colour set $C - C' \cup \{C'\}$ by retaining the colours for points whose colours do not lie in C', and by colouring non-trivial $\sim_2$-classes by C' itself (which is now regarded as a single colour).

Lemma 2.7 *For any countable dense C-coloured linear ordering A, and non-empty subset C' of C which is minimal subject to $\sim_2$ being a non-trivial equivalence relation, each $\sim_2$-class is isomorphic to $\mathbb{Q}_{C'}$ (possibly with endpoints adjoined).*

P r o o f. Let $x < y$ lie in a $\sim_2$-class of A, and suppose for a contradiction that for some $c \in C'$, no point of (x, y) is coloured c. Since A is dense, (x, y) is non-empty, so $C' - \{c\}$ is also non-empty. Let $\sim_2'$ be the equivalence relation obtained from $C' - \{c\}$ in place of C'. Then there is a non-trivial equivalence class, since all points of (x, y) are $\sim_2'$-equivalent, contrary to minimality of C'. Hence each colour in C' occurs densely throughout the $\sim_2$-class, and by back-and-forth it is isomorphic to $\mathbb{Q}_{C'}$ (possibly with endpoints adjoined). □

Lemma 2.8 *Any countable $\mathbb{Q}_n$-combination or concatenation of finitely many $\aleph_0$-categorical coloured linear orderings is also $\aleph_0$-categorical.*

P r o o f. For concatenations it suffices to do this for two orderings, and then repeat. So we let A and B be disjoint $\aleph_0$-categorical coloured linear orderings; we need to show that $A \cup B$ is also an $\aleph_0$-categorical coloured linear order where $A < B$. Let a_1 and a_2 be the numbers of 1-orbits (1-types in our case) and 2-orbits of pairs $\langle x, y\rangle$ in A with $x < y$ and let b_1 and b_2 be the similar numbers for B. Thus $A \cup B$ has at most $a_1 + b_1$ 1-orbits. By Lemma 2.3 it thus suffices to consider the 2-orbits of pairs $\langle x, y\rangle$ in $A \cup B$ with $x < y$. The list of orbits of such pairs in $A \cup B$ then comprises a_2 2-orbits on A, b_2 2-orbits on B and $a_1 b_1$ in the case where $x \in A$ and $y \in B$ ($a_1 b_1$ is precisely the number of 2-orbits on $A \times B$). Therefore the corresponding number of 2-orbits on $A \cup B$ is at most $a_2 + b_2 + a_1 b_1$. By Lemma 2.3 and the Engeler-Ryll–Nardzewski-Svenonius Theorem, $A \cup B$ is $\aleph_0$-categorical.

Now consider a $\mathbb{Q}_n$-combination of coloured orderings $A_0, A_1, \dots, A_{n-1}$ where each A_i has a_i' 1-orbits and a_i 2-orbits on increasing pairs. By the same method as above, we see that $\mathbb{Q}_n(A_0, A_1, \dots, A_{n-1})$ has $\sum_{i=0}^{n-1} a_i'$ 1-orbits and at most $\sum_{i=0}^{n-1} a_i + \sum_{i,j} a_i' a_j'$ 2-orbits (using the fact that $\mathbb{Q}_n$ is itself homogeneous). Once again we conclude by appealing to Lemma 2.3. □

We can now give our main theorem.

Theorem 2.9 *A finite or countable coloured linear ordering $(A, <, F)$ is $\aleph_0$-categorical if and only if it can be built up in finitely many steps from coloured singletons by using concatenation and $\mathbb{Q}_n$-combinations for finite $n \geq 1$.*

P r o o f. In one direction this follows immediately from Lemma 2.8 and the fact that any coloured singleton is trivially $\aleph_0$-categorical.

Conversely, suppose that $(A, <, F)$ is countable and $\aleph_0$-categorical. We use $\sim_1$ and $\sim_2$ to analyze A. We define A_n having colour set C_n inductively thus, so that each A_n is a partition of A into convex subsets. Let $A = A_0$ (or strictly speaking, $\{\{x\} : x \in A\}$) and $C_0 = C$. Suppose that A_n and C_n have been defined. If $\sim_1$ is non-trivial on A_n (which means that it is not a singleton, and is not dense), then we let $A_{n+1} = A_n/\sim_1$ (strictly speaking, the partition of A induced by this) with the colour set C_{n+1} obtained as after the proof of Lemma 2.2 from C_n. If A_n is dense, then there is some choice of non-empty subset C' of C_n such that $\sim_2$ is non-trivial on A_n (for instance, C_n itself), and we choose one such C' of least size, and let $A_{n+1} = A_n/\sim_2$ (again with the naturally induced colouring, so that $C_{n+1} = C_n - C' \cup \{C'\}$). Thus as n increases we are identifying more and more elements of A.

We shall show that the sequence $A_0, A_1, A_2, \dots$ terminates in finitely many steps. Note that the elements of A_{n+1} are convex subsets of A_n and hence the elements of A_n for each n are convex subsets of A. Choose any $x \in A$, and let $(x)_n$ be the element of A_n that x belongs to. Thus $\{x\} = (x)_0 \subseteq (x)_1 \subseteq (x)_2 \subseteq \cdots$. Whenever

$(x)_n \neq (x)_{n+1}$ we pick an element y_n of $(x)_{n+1} - (x)_n$. Now each A_n is defined from $(A, <, F)$, so is fixed by its automorphism group. Hence the stabilizer of x fixes each $(x)_n$ (setwise), and so if $m < n$ and y_m and y_n are both defined, since $y_m \in (x)_n$ and $y_n \notin (x)_n$, they are in different orbits of the stabilizer of x. Hence the pairs $\langle x, y_m \rangle$ and $\langle x, y_n \rangle$ have different 2-types. As there are only finitely many 2-types, y_n is defined for only finitely many n, and so the chain $\{x\} = (x)_0 \subseteq (x)_1 \subseteq (x)_2 \subseteq \cdots$ is eventually constant. Let the first point at which it becomes constant be $n(x)$. Since there are only finitely many 1-types, as x varies, there are only finitely many values of $n(x)$, and hence the sequence $A_0, A_1, A_2, \ldots$ terminates after finitely many steps, at A_N say. It follows that A_N is a singleton.

Now we can show by induction on N that A is built up in finitely many steps from coloured singletons by concatenation and $\mathbb{Q}_n$-combinations. For $N = 0$ this is clear, since then A is a coloured singleton. Otherwise, $A_1 = A/{\sim_1}$ or $A/{\sim_2}$. Consider the first case. By definition, the colours on $A/{\sim_1}$ are given by the orbits of $\sim_1$-classes of A, and there are just finitely many of these. By induction hypothesis, since the sequence starting from A_1 has length just $N - 1$, A_1 can be built up from coloured singletons by concatenation and $\mathbb{Q}_n$-combinations, where these colours are in C_1. But each of these 'colours' in C_1 is built up from coloured singletons by concatenation, and putting these together gives a way of building up A itself from coloured singletons in C. If however $A_1 = A/{\sim_2}$, where $C' \subseteq C_1$ is the (minimal) set of colours used in defining $\sim_2$, then by Lemma 2.7 A is obtained from A_1 by replacing each point of colour C' by a copy of $\mathbb{Q}_{C'}$, possibly with one or both endpoint, and so in this case too, A is built up from coloured singletons by concatenation and $\mathbb{Q}_n$-combinations (since if one or both endpoint is required, they can again be adjoined by concatenation).

This process will be elucidated in the next section by the use of coding trees. □

3 Coding trees

We conclude by giving a natural tree representation for the structures in our class, which relates them to the 1-transitive (finitely-)coloured linear orders studied in [1]. Here, a *tree* is a finite partially ordered set $(\chi, \prec)$ such that for every $x, y \in \chi$, there is a $z \in \chi$ with $x \preceq z$ and $y \preceq z$ and such that for every $x \in \chi$, $\{y \in \chi : x \preceq y\}$ is linearly ordered. The greatest element of a tree is called its *root*, r; and the minimal elements are called the *leaves*. The *levels* of a tree tell us the number of steps required to backtrack to the root.

If $x, y \in \chi$, we say that y is a *child* of x or x is a *parent* of y and write $y \prec\!\prec x$, if $y \prec x$ and there is no $z \in \chi$ such that $y \prec z \prec x$. Distinct children of the same parent are called *siblings*. A *labelled tree* $(\chi, \prec, \xi)$ is a tree $(\chi, \prec)$, together with a function $\xi : \chi \longrightarrow L$, where L is a set of 'labels'. The labels will be ordered pairs $\xi(x) = (\varphi(x), \eta(x))$, where $\varphi(x)$ (except at leaves) tells us how the coloured linear ordering associated with that vertex is constructed from those associated with its children, and $\eta(x)$ tells us what the colour set is for that coloured linear order. The ordered pair at a leaf is $(1, \{c\})$ for some $c \in C$.

A *coding tree* has the form $(\chi, \prec, \xi, \psi)$, where $(\chi, \prec, \xi)$ is a labelled tree in the above sense with every label an ordered pair, and:

(a) ψ is a linear ordering of the branches of χ induced by a linear ordering of the children of each vertex.

(b) If $x \prec y$ and $y \neq r$, then either x has a sibling or y has a sibling.

(c) If x has only one child, then the first entry $\varphi(x)$ of its label is $\mathbb{Q}_1$.

(d) If x has $n \geq 2$ children, then $\varphi(x)$ is $\mathbb{Q}_n$ or n.

(e) If $x \neq r$ and x is not a leaf, then at least one of the following holds:

(i) it has no sibling, and $\varphi(x) = n$, where n is its number of children;

(ii) it has a sibling, and only one child, and $\varphi(x) = \mathbb{Q}_1$;

(iii) the first label of its parent is n, for $2 \leq n < \omega$ and $\varphi(x) = \mathbb{Q}_m$, for some $m \geq 2$;

(iv) x has a sibling and $n \geq 2$ children and the first label of its parent is $\mathbb{Q}_m$ with $2 \leq m < \omega$ and $\varphi(x) = n$;

(v) the first label of its parent is k and it has a sibling labelled $\mathbb{Q}_m$ and $\varphi(x) = n$ where $2 \leq k, n < \omega$ and $1 \leq m < \omega$.

(f) If $x \in \chi$ is a leaf, then $\varphi(x) = 1$.

(g) The second member $\eta(x)$, of the label at x is a subset of the set of colours C such that $\eta(r) = C$, if x is a leaf, then $\eta(x)$ is a singleton, and if x is not a leaf, then $\eta(x)$ is the union of $\eta(y)$ for the children y of x.

A coding tree $(\chi, \prec, \xi, \psi)$ *encodes* the coloured linear order $(A, <, F)$ if we can assign coloured linear orders to the vertices of χ by a function f in such a way that $f(r) = (A, <, F)$, for each x the colours occurring in $f(x)$ are the members of $\eta(x)$, a leaf with second co-ordinate c is assigned a singleton linear order with that colour, and if x is a non-leaf vertex, $f(x)$ is obtained from $\{f(y)\colon y \text{ a child of } x\}$ according to $\varphi(x)$ as follows:

If $\varphi(x)$ is $\mathbb{Q}_n$, where $1 \leq n < \omega$, and $y_0, y_1, \ldots, y_{n-1}$ are its children in the order given by ψ, then $f(x) = \mathbb{Q}_n(f(y_0), f(y_1), \ldots, f(y_{n-1}))$, if $\varphi(x)$ is n, where $2 \leq n < \omega$, and $y_0, y_1, \ldots, y_{n-1}$ are its children in the order given by ψ, then $f(x)$ is the concatenation of $f(y_0), f(y_1), \ldots, f(y_{n-1})$.

We remark that in [1], it was demanded that the children of a particular vertex have disjoint colour sets (in view of the 1-transitivity condition). However, for us the children can have intersecting colour sets. The main case in [1] which is absent here is the lexicographic product with a general countable 1-transitive linear order Z. Because of $\aleph_0$-categoricity, the only instance of this which survives here is where $Z = \mathbb{Q} = \mathbb{Q}_1$, and we may if we wish view the lexicographic product as $\mathbb{Q}_1(f(y_0))$, thus fitting in with the general $\mathbb{Q}_n$ clause.

Theorem 3.1 *Any coding tree χ encodes a coloured linear ordering and this is a finite or countable $\aleph_0$-categorical coloured linear ordering.*

P r o o f. It follows from the definition of 'coloured linear order encoded' by a coding tree, that the coloured linear order encoded by χ can be built up from coloured singletons using concatenations and $\mathbb{Q}_n$-combinations in finitely many steps, and hence by Theorem 2.9 is $\aleph_0$-categorical. □

Theorem 3.2 *Any finite or countable $\aleph_0$-categorical coloured linear ordering $(A, <, F)$ can be represented by a coding tree.*

P r o o f. By Theorem 2.9, A can be built up in finitely many steps from coloured singletons using concatenation and $\mathbb{Q}_n$-combinations. This enables us to give an inductive construction of the tree. For a coloured singleton we use a tree with one element labelled $(1, \{c\})$ where A is coloured c. Otherwise, if A is built up as a $\mathbb{Q}_n$-combination from $A_0, \ldots, A_{n-1}$, each A_i is built up in fewer steps than A, so we have coding trees for each of these by induction hypothesis, and the coding tree for A has label $(\mathbb{Q}_n, C)$ at the root, with n children, which have the coding trees for $A_0, \ldots, A_{n-1}$ below them. We treat the case of concatenation similarly. □

In summary, we can say that the list of structures we have presented here differs from Rosenstein's in that we can start with coloured singletons, rather than just singletons. Compared with the coloured orders given in [1] and [2], the ones here are richer in that in performing the two operations of concatenation and $\mathbb{Q}_n$-combinations, we do not require the orderings that we are combining to have disjoint colour sets, but they are more restricted in the sense that the set of colours has to be finite.

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