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Abstract

In this paper we present quantitative upper estimates for the number of negative eigenvalues of a two-dimensional Schrödinger operator with potential supported by an unbounded Lipschitz curve. The estimates are given in terms of weighted L^1 and $L \log L$ type Orlicz norms of the potential.

Keywords: Negative eigenvalues; Schrödinger operators; Lipschitz curves.

1 Introduction

According to the celebrated Cwikel-Lieb-Rozenblum inequality [2, 16], the number $N_-(V)$ of negative eigenvalues of the Schrödinger operator $-\Delta - V$, $V \geq 0$ on $L^2(\mathbb{R}^d)$, $d \geq 3$ is estimated above by

$$\text{const} \int_{\mathbb{R}^d} V(x)^{\frac{d}{2}}\, dx.$$

It is well known that this estimate does not hold for $d = 2$. One of the reasons for this is that the Sobolev space $H^1(\mathbb{R}^2)$ is not continuously embedded in $L^\infty(\mathbb{R}^2)$. However, it was shown in [9] that the Cwikel-Lieb-Rozenblum inequality gives a lower estimate in two dimensions. The strongest known estimates for the number of negative eigenvalues of a two-dimensional Schrödinger

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operator involve weighted L^1 norms and $L \log L$ type Orlicz norms of the potential (see, e.g., [18, 19]). In the case of compactly supported potentials, one does not need to use weighted L^1 norms of the potential. Similar estimates involving only $L \log L$ Orlicz norms of the potential were obtained in [17] for two-dimensional Schrödinger operators with potentials supported by bounded Lipschitz curves. The estimates obtained in [17] are extended in the present paper to the case of potentials supported by unbounded Lipschitz curves. This extension is by no means straightforward as one needs to introduce weighted L^1 norms of the potential into the estimates. More upper estimates for $N_-(V)$ in the case $d = 2$ can be found in [4, 6, 7, 8, 13, 18, 19] and the references therein.

We study the operator

$$H_V := -\Delta - V, \quad V \geq 0 \tag{1}$$

on $L^2(\mathbb{R}^2)$, where V is a real valued function supported by and locally integrable on an unbounded Lipschitz curve.

2 Notation and auxilliary material

In this section we present the basic theory of Orlicz spaces that we use in the sequel (more details can be found for example in [1, 12] and [15]).

Let (Ω, Σ, μ) be a measure space and let $\Psi : [0, +\infty) \to [0, +\infty)$ be a non-decreasing function. We define a Banach space of functions $f : \Omega \to \mathbb{C}$ (or $\mathbb{R}$) such that

$$\int_\Omega \Psi\big(|f(x)|\big) d\mu(x) < \infty. \tag{2}$$

If $\Psi(t) \equiv t^p$, then this is just $L^p(\Omega)$. If Ψ is rapidly increasing, e.g exponentially increasing, the set of all functions satisfying (2) is not a linear space **as** (2) does not imply that the same for $2f$ is finite.

Definition 2.1. A continuous convex non-decreasing function $\Psi : [0, +\infty) \to [0, +\infty)$ is called an N-function if

$$\lim_{t \to 0+} \frac{\Psi(t)}{t} = 0 \qquad \text{and} \qquad \lim_{t \to +\infty} \frac{\Psi(t)}{t} = +\infty.$$

The function $\Phi : [0, +\infty) \to [0, +\infty)$ defined by

$$\Phi(t) := \sup_{s \geq 0} \big(st - \Psi(s)\big)$$

is called complementary to Ψ.

Let Φ and Ψ be mutually complementary N-functions. We will use the following norms:

$$\|f\|_{\Psi,\Omega} = \sup\left\{\left|\int_\Omega fg d\mu\right| : \int_\Omega \Phi(|g|)d\mu \le 1\right\} \tag{3}$$

and

$$\|f\|_{(\Psi,\Omega)} = \inf\left\{\kappa > 0 : \int_\Omega \Psi\left(\frac{|f|}{\kappa}\right)d\mu \le 1\right\}. \tag{4}$$

The Orlicz space which we denote by $L_\Psi(\Omega)$ is defined as follows

$$L_\Psi(\Omega) := \{f \mid \|f\|_{\Psi,\Omega} < \infty\}. \tag{5}$$

The above two norms are equivalent

$$\|f\|_{(\Psi,\Omega)} \le \|f\|_{\Psi,\Omega} \le 2\|f\|_{(\Psi,\Omega)}, \quad \forall f \in L_\Psi(\Omega), \tag{6}$$

(see [1]).
Note that

$$\int_\Omega \Psi\left(\frac{|f|}{\kappa_0}\right)d\mu \le C_0, \quad C_0 \ge 1 \implies \|f\|_{(\Psi,\Omega)} \le C_0\kappa_0. \tag{7}$$

Indeed, since Ψ is convex and increasing on $[0,+\infty)$, and $\Psi(0) = 0$, we get for any $\kappa \ge C_0\kappa_0$,

$$\int_\Omega \Psi\left(\frac{|f|}{\kappa}\right)d\mu \le \int_\Omega \Psi\left(\frac{|f|}{C_0\kappa_0}\right)d\mu \le \frac{1}{C_0}\int_\Omega \Psi\left(\frac{|f|}{\kappa_0}\right)d\mu \le 1. \tag{8}$$

It follows from (7) with $\kappa_0 = 1$ that

$$\|f\|_{(\Psi,\Omega)} \le \max\left\{1, \int_\Omega \Psi(|f|)d\mu\right\}. \tag{9}$$

We will also need the following equivalent norm on $L_\Psi(\Omega)$ with $\mu(\Omega) < \infty$ which was introduced in [19]:

$$\|f\|^{(av)}_{\Psi,\Omega} := \sup\left\{\left|\int_\Omega fg\, d\mu\right| : \int_\Omega \Phi(|g|)d\mu \le \mu(\Omega)\right\}. \tag{10}$$

Let

$$\|f\|^{(av),\tau}_{\Psi,\Omega} = \sup\left\{\left|\int_\Omega fg\, d\mu\right| : \int_\Omega \Phi(|g|)d\mu \le \tau\mu(\Omega)\right\}, \quad \tau > 0. \tag{11}$$

By mimicking the proof of Lemma 2.1 in [18], one can show that for any $\tau_1, \tau_2 > 0$

$$\min\left\{1, \frac{\tau_2}{\tau_1}\right\}\|f\|^{(av),\tau_1}_{\Psi,\Omega} \le \|f\|^{(av),\tau_2}_{\Psi,\Omega} \le \max\left\{1, \frac{\tau_2}{\tau_1}\right\}\|f\|^{(av),\tau_1}_{\Psi,\Omega}. \tag{12}$$

Lemma 2.2. Let Ω_k, $k = 1, ..., n$ be pairwise disjoint measurable subsets of Ω such that $\underset{k}{\cup}\Omega_k \subseteq \Omega$. Then

$$\sum_{k=1}^{n} \|f\|^{(\mathrm{av})}_{\Psi,\Omega_k} \le \|f\|^{(\mathrm{av})}_{\Psi,\Omega}.$$

Let G, G_k, $k = 1, ..., n$ be measurable sets such that $G \subseteq \underset{k}{\cup} G_k$. Then

$$\|f\|^{(\mathrm{av})}_{\Psi,G} \le \beta \sum_{k=1}^{n} \|f\|^{(\mathrm{av})}_{\Psi,G_k}\,, \tag{13}$$

where

$$\beta := \max\left\{1, \max_{1,2,...,n} \frac{\mu(G)}{\mu(G_k)}\right\}.$$

Proof. The first statement can be found in [19, Lemma 2]. Now

$$\begin{aligned}
\|f\|^{(av)}_{\Psi,G} &= \sup\left\{\left|\int_G fg\,d\mu\right| : \int_G \Phi(|g|)d\mu \le \mu(G)\right\} \\
&\le \sup\left\{\sum_{k=1}^{n}\left|\int_{G\cap G_k} fg\,d\mu\right| : \int_G \Phi(|g|)d\mu \le \mu(G)\right\} \\
&\le \sum_{k=1}^{n} \sup\left\{\left|\int_{G_k} fg\,d\mu\right| : \int_{G_k} \Phi(|g|)d\mu \le \beta\mu(G_k)\right\} \\
&\le \sum_{k=1}^{n} \|f\|^{(\mathrm{av}),\beta}_{\Psi,G_k} \\
&\le \beta\sum_{k=1}^{n} \|f\|^{(\mathrm{av})}_{\Psi,G_k}
\end{aligned}$$

(by (12)). □

We will use the following pair of mutually complementary N-functions

$$\mathcal{A}(s) = e^{|s|} - 1 - |s|, \quad \mathcal{B}(s) = (1 + |s|)\ln(1 + |s|) - |s|, \quad s \in \mathbb{R}. \tag{14}$$

3 Lipschitz curves

Definition 3.1. A mapping F from a metric space (X_1, d_1) into a metric space (X_2, d_2) is said to bi-Lipschitz if there exists a constant $L > 0$ such that

$$d_1(x, y)/L \le d_2(F(x), F(y)) \le Ld_1(x, y), \quad \forall x, y \in X_1.$$

Let ℓ be a curve in $\mathbb{C}$. We say that ℓ is a Lipschitz arc if it is a bi-Liptschitz image of $[0, 1]$.

It is clear that a Lipschitz arc is non-self intersecting and rectifiable (has finite length). Using arc length parametrization, one is able to see that a simple rectifiable curve ℓ is a Lipschitz arc if and only if it is a chord-arc curve, that is, there exists a constant $K \geq 1$ such that the length of the sub-arc of ℓ joining any two points is bounded by K times the distance between them. When dealing with function spaces on ℓ, we will always assume that ℓ is equipped with the arc length measure.

Let Γ be an unbounded curve in $\mathbb{R}^2$ and $F : \mathbb{R} \longrightarrow \mathbb{R}^2$ be a bi-Lipschitz mapping such that $F(\mathbb{R}) = \Gamma$. Then F can be extended to a bi-Lipschitz homeomorphism $F : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ (see [11, Proposition 1.13]). Below we identify $\mathbb{R}^2$ with $\mathbb{C}$. Let $\xi_0 \in \mathbb{C} \setminus \mathbb{R}$. Then one has

$$\begin{aligned}
&\int_\Gamma \ln(1 + |z - z_0|) \frac{1}{|z - z_0|^2} ds(z) \\
&= \int_\mathbb{R} \ln(1 + |F(\xi) - F(\xi_0)|) \frac{1}{|F(\xi) - F(\xi_0)|^2} dF(\xi) \\
&\leq \int_\mathbb{R} \ln(1 + L|\xi - \xi_0|) \frac{L\, d\xi}{|\xi - \xi_0|^2/L^2} \\
&= L^3 \int_\mathbb{R} \ln(1 + L|\xi - \xi_0|) \frac{d\xi}{|\xi - \xi_0|^2} < \infty.
\end{aligned} \tag{15}$$

Lemma 3.2. Let $z_0 \in \mathbb{C} \setminus \Gamma$ be fixed and let

$$\xi(z) := \frac{1}{z - z_0}, \quad z \in \mathbb{C} \setminus \{z_0\}. \tag{16}$$

Then $\tilde{\Gamma} := \xi(\Gamma) \cup \{0\}$ is a closed Lipschitz curve, i.e. a bi-Liptschitz image of the unit circle.

Proof. Let $\mathbb{T} := \{\zeta \in \mathbb{C} : \ |\zeta| = 1\}$ and

$$\begin{aligned}
\omega(\zeta) &:= i\, \frac{\zeta - 1}{\zeta + 1}, \quad \zeta \in \mathbb{C} \setminus \{-1\}, \\
F_0(\zeta) &:= \xi(F(\omega(\zeta))) = \frac{1}{F(\omega(\zeta)) - z_0}\,.
\end{aligned}$$

Then $\xi(\Gamma) = \xi(F(\mathbb{R})) = \xi(F(\omega(\mathbb{T} \setminus \{-1\}))) = F_0(\mathbb{T} \setminus \{-1\})$ and $F_0(\zeta) \to 0$ as $\zeta \to -1$.

There exists $w_0 \in \mathbb{C} \setminus \mathbb{R}$ such that $z_0 = F(w_0)$. For any $\zeta_1, \zeta_2 \in \mathbb{T} \setminus \{-1\}$, we have

$$\begin{aligned} F_0(\zeta_1) - F_0(\zeta_2) &= \frac{1}{F(\omega(\zeta_1)) - F(w_0)} - \frac{1}{F(\omega(\zeta_2)) - F(w_0)} \\ &= \frac{F(\omega(\zeta_2)) - F(\omega(\zeta_1))}{(F(\omega(\zeta_1)) - F(w_0))\,(F(\omega(\zeta_2)) - F(w_0))}\,. \end{aligned} \tag{17}$$

Now

$$\begin{aligned} \frac{2|\zeta_2 - \zeta_1|}{L|\zeta_2 + 1||\zeta_1 + 1|} = \frac{1}{L}|\omega(\zeta_2) - \omega(\zeta_1)| &\le |F(\omega(\zeta_j)) - F(\omega(\zeta_1))| \\ &\le L|\omega(\zeta_2) - \omega(\zeta_1)| = 2L\frac{|\zeta_2 - \zeta_1|}{|\zeta_2 + 1||\zeta_1 + 1|} \end{aligned} \tag{18}$$

and

$$\begin{aligned} \frac{|\zeta_j(iw_0 + 1) + (iw_0 - 1)|}{L|\zeta_j + 1|} &= \frac{1}{L}\,|\omega(\zeta_j) - w_0| \le |F(\omega(\zeta_j)) - F(w_0)| \\ \le L\,|\omega(\zeta_j) - w_0| &= L\,\frac{|\zeta_j(iw_0 + 1) + (iw_0 - 1)|}{|\zeta_j + 1|}\,, \quad j = 1, 2. \end{aligned} \tag{19}$$

Since $w_0 \notin \mathbb{R}$, we have

$$M := \max_{\zeta \in \mathbb{T}} |\zeta(iw_0 + 1) + (iw_0 - 1)| < +\infty,$$

$$m := \min_{\zeta \in \mathbb{T}} |\zeta(iw_0 + 1) + (iw_0 - 1)| > 0.$$

It now follows from (17), (18) and (19) that

$$\frac{2|\zeta_1 - \zeta_2|}{L^3 M^2} \le |F_0(\zeta_1) - F_0(\zeta_2)| \le \frac{2L^3}{m^2}|\zeta_1 - \zeta_2|.$$

Hence $F_0 : \mathbb{T}\setminus\{-1\} \to \xi(\Gamma)$ is bi-Lipschitz and can be extended by continuity to a bi-Lipschitz mapping from $\mathbb{T}$ onto $\tilde{\Gamma}$. □

4 Estimates for the number of negative eigenvalues

Let $\mathcal{H}$ be a Hilbert space and let $\mathbf{q}$ be a Hermitian form with a domain $\mathrm{Dom}\,(\mathbf{q}) \subseteq \mathcal{H}$. Set

$$N_-(\mathbf{q}) := \sup\left\{\dim \mathcal{L} \mid \mathbf{q}[u] < 0,\ \forall u \in \mathcal{L} \setminus \{0\}\right\}, \tag{20}$$

where $\mathcal{L}$ denotes a linear subspace of Dom $(\mathbf{q})$. If $\mathbf{q}$ is the quadratic form of a self-adjoint operator A with no essential spectrum in $(-\infty, 0)$, then, $N_-(\mathbf{q})$ is the number of negative eigenvalues of A repeated according to their multiplicity (see, e.g., [3, S1.3] or [5, Theorem 10.2.3]).

Let ℓ be a Lipschitz arc in $\mathbb{R}^2$. Denote by $N_-(V)$ the number of negative eigenvalues counting multiplicities of a two dimensional Schrödinger operator with potential V supported by ℓ.

Theorem 4.1. [17, Theorem 3.1] Let $V \in L_{\mathcal{B}}(\ell)$ (see (5) and (14)). Then there exists a constant $C(\ell) > 0$ such that

$$N_-(V) \leq C(\ell)\|V\|_{\mathcal{B},\ell} + 1, \quad \forall V \in L_{\mathcal{B}}(\ell),\, V \geq 0. \tag{21}$$

Below, Γ and $\tilde{\Gamma}$ are as defined in §3 and $W_2^1(\mathbb{R}^2)$ denotes the standard Sobolev space $H^1(\mathbb{R}^2)$. Let $V \in L^1_{\text{loc}}(\Gamma)$ and define the operator (1) via its quadratic form by

$$\begin{aligned} q_V[w] &:= \int_{\mathbb{R}^2} |\nabla w(z)|^2\, dz - \int_\Gamma V(z)|w(z)|^2\, ds(z), \\ \text{Dom}(q_V) &= W_2^1(\mathbb{R}^2) \cap L^2(\Gamma, V\, ds). \end{aligned}$$

We shall denote by $N_-(q_V)$ the number of negative eigenvalues of (1) counting multiplicities.

Theorem 4.2. Let $z_0 \in \mathbb{C} \setminus \Gamma$ such that $\inf_{z\in\Gamma}|z - z_0| \geq 1$. Then for all $V \in L_{\mathcal{B}}(\Gamma)$, $V \geq 0$, there exists a constant $C_1(\Gamma) > 0$ such that

$$N_-(q_V) \leq C_1(\Gamma)\left(\|V\|_{\mathcal{B},\Gamma} + \int_\Gamma V(z) \ln(1 + |z - z_0|)\, ds(z)\right) + 1. \tag{22}$$

Proof. Let $\tilde{w}(\xi) := w(z)$, $\tilde{V}(\xi) := |z - z_0|^2 V(z)$, where ξ is given by (16). Then

$$\begin{aligned} \int_{\mathbb{R}^2} |\nabla w(z)|^2\, dz &= \int_{\mathbb{R}^2} |\nabla \tilde{w}(\xi)|^2\, d\xi \\ \int_\Gamma V(z)|w(z)|^2\, ds(z) &= \int_{\tilde{\Gamma}} \tilde{V}(\xi)|\tilde{w}(\xi)|^2\, ds(\xi). \end{aligned}$$

Let

$$\begin{aligned} q_{\tilde{V}}[\tilde{w}] &:= \int_{\mathbb{R}^2} |\nabla \tilde{w}(\xi)|^2\, d\xi - \int_{\tilde{\Gamma}} \tilde{V}|\tilde{w}(\xi)|^2\, ds(\xi), \\ \text{Dom}(q_{\tilde{V}}) &= W_2^1(\mathbb{R}^2) \cap L^2(\tilde{\Gamma}, \tilde{V}\, ds). \end{aligned}$$

Then by (21) we have

$$N_-(q_V) = N_-(q_{\tilde{V}}) \le C(\tilde{\Gamma})\|\tilde{V}\|_{\mathcal{B},\tilde{\Gamma}} + 1, \quad \tilde{V} \ge 0, \tag{23}$$

It now remains to estimate the norm in the right-hand side (23). For convenience, we work with the Luxemburg norm (4). For any $\kappa > 0$, we have

$$\begin{aligned}
& \int_{\tilde{\Gamma}} \mathcal{B}\left(\frac{\tilde{V}(\xi)}{\kappa}\right) ds(\xi) = \int_{\tilde{\Gamma}} \mathcal{B}\left(\frac{V(z)}{\kappa|\xi|^2}\right) ds(\xi) \\
= & \int_{\Gamma} \mathcal{B}\left(\frac{V(z)|z-z_0|^2}{\kappa}\right) \frac{1}{|z-z_0|^2}\, ds(z) \\
= & \int_{\Gamma} \left(\left(1 + |z-z_0|^2\frac{V(z)}{\kappa}\right) \ln\left(1 + |z-z_0|^2\frac{V(z)}{\kappa}\right) - |z-z_0|^2\frac{V(z)}{\kappa}\right) \frac{1}{|z-z_0|^2}\, ds(z) \\
\le & \int_{\Gamma} \left(\left(1 + |z-z_0|^2\frac{V(z)}{\kappa}\right) \ln\left(1 + \frac{V(z)}{\kappa}\right) - |z-z_0|^2\frac{V(z)}{\kappa}\right) \frac{1}{|z-z_0|^2}\, ds(z) \\
+ & \int_{\Gamma} \left(\left(1 + |z-z_0|^2\frac{V(z)}{\kappa}\right) \ln\left(1 + |z-z_0|^2\right)\right) \frac{1}{|z-z_0|^2}\, ds(z) \\
= & \int_{\Gamma} |z-z_0|^2 \left(\left(\frac{1}{|z-z_0|^2} + \frac{V(z)}{\kappa}\right) \ln\left(1 + \frac{V(z)}{\kappa}\right) - \frac{V(z)}{\kappa}\right) \frac{1}{|z-z_0|^2}\, ds(z) \\
+ & \int_{\Gamma} \left(\left(1 + |z-z_0|^2\frac{V(z)}{\kappa}\right) \ln\left(1 + |z-z_0|^2\right)\right) \frac{1}{|z-z_0|^2}\, ds(z) \\
\le & \max\left\{1, \sup_{z\in\Gamma}\frac{1}{|z-z_0|^2}\right\} \int_{\Gamma} \left(\left(1 + \frac{V(z)}{\kappa}\right) \ln\left(1 + \frac{V(z)}{\kappa}\right) - \frac{V(z)}{\kappa}\right) ds(z) \\
+ & \frac{1}{\kappa} \int_{\Gamma} V(z) \ln\left(1 + |z-z_0|^2\right) ds(z) + \int_{\Gamma} \ln\left(1 + |z-z_0|^2\right) \frac{1}{|z-z_0|^2}\, ds(z) \\
\le & \int_{\Gamma} \mathcal{B}\left(\frac{V(z)}{\kappa}\right) ds(z) + \frac{2}{\kappa}\int_{\Gamma} V(z) \ln\left(1 + |z-z_0|\right) ds(z) \\
+ & 2\int_{\Gamma} \ln\left(1 + |z-z_0|\right) \frac{1}{|z-z_0|^2}\, ds(z).
\end{aligned}$$

Let

$$\kappa_0 := \max\left\{\|V\|_{(\mathcal{B},\Gamma)}, \int_{\Gamma} V(z) \ln\left(1 + |z-z_0|\right) ds(z)\right\}.$$

Then

$$\int_{\tilde{\Gamma}} \mathcal{B}\left(\frac{\widetilde{V}(\xi)}{\kappa_0}\right) ds(\xi) \le 1 + 2 + 2\int_{\Gamma} \ln\left(1 + |z-z_0|\right) \frac{1}{|z-z_0|^2}\, ds(z) := C_2 < \infty$$

(by (15)). Hence, by (7) we have

$$\begin{aligned}\|\widetilde{V}\|_{(\mathcal{B},\widetilde{\Gamma})} &\le C_2 \max\left\{\|V\|_{(\mathcal{B},\Gamma)}, \int_\Gamma V(z)\ln(1+|z-z_0|)\, ds(z)\right\} \\ &\le C_2\left(\|V\|_{(\mathcal{B},\Gamma)} + \int_\Gamma V(z)\ln(1+|z-z_0|)\, ds(z)\right).\end{aligned}$$

By (6) and (23) the proof is complete.

□

In the next theorem, we give an analogue of the Birman-Solomyak type estimate presented in [6] (see also §5 in [18]). Without loss of generality, assume that $F(0)=0$. Let

$$\begin{aligned}
U_n &:= [e^{2^{n-1}}, e^{2^n}],\ n>0, \quad U_0 := [e^{-1}, e], \quad U_n := [e^{-2^{|n|}}, e^{-2^{|n|-1}}],\ n<0,\\
\Lambda_n &:= \left\{x\in\mathbb{R}^2 \ :\ \frac{1}{L}e^{2^{n-1}} \le |x| \le Le^{2^n}\right\}\cap\Gamma,\ n>0,\\
\Lambda_0 &:= \left\{x\in\mathbb{R}^2 \ :\ \frac{1}{L}e^{-1} \le |x| \le Le\right\}\cap\Gamma,\\
\Lambda_n &:= \left\{x\in\mathbb{R}^2 \ :\ \frac{1}{L}e^{-2^{|n|}} \le |x| \le Le^{-2^{|n|-1}}\right\}\cap\Gamma,\ n<0,\\
\gamma_n &:= \left\{x\in\mathbb{R}^2 \ :\ e^{2^{n-1}} \le |x| \le e^{2^n}\right\}\cap\Gamma,\ n>0,\\
\gamma_0 &:= \left\{x\in\mathbb{R}^2 \ :\ e^{-1} \le |x| \le e\right\}\cap\Gamma,\\
\gamma_n &:= \left\{x\in\mathbb{R}^2 \ :\ e^{-2^{|n|}} \le |x| \le e^{-2^{|n|-1}}\right\}\cap\Gamma,\ n<0.
\end{aligned}$$

There exists an integer n_0 such that $e^{2^{n_0-1}} \ge L$. One can take $n_0 = \lceil \log_2 \log L\rceil + 1$, where $\lceil\cdot\rceil$ denotes the ceiling function, i.e., $\lceil a\rceil$ is the least integer greater than or equal to a. Increasing n_0 if necessary, we can assume that $n_0 \ge 0$. Thus

$$\begin{aligned}
\Lambda_n &\subset \left\{x\in\mathbb{R}^2 \ :\ e^{2^{n-1}-2^{n_0-1}} \le |x| \le e^{2^n+2^{n_0-1}}\right\},\quad n>0,\\
\Lambda_0 &\subset \left\{x\in\mathbb{R}^2 \ :\ e^{-1-2^{n_0-1}} \le |x| \le e^{1+2^{n_0-1}}\right\},\\
\Lambda_n &\subset \left\{x\in\mathbb{R}^2 \ :\ e^{-2^{|n|}-2^{n_0-1}} \le |x| \le e^{-2^{|n|-1}+2^{n_0-1}}\right\},\quad n<0.
\end{aligned}$$

Let

$$n_+ := \begin{cases} n+1, & |n|>n_0\\ n_0+1, & |n|\le n_0,\end{cases} \qquad n_- := \begin{cases} n-1, & |n|>n_0\\ -n_0-1, & |n|\le n_0.\end{cases} \tag{24}$$

If $n > n_0$, then $2^{n_+} = 2^{n+1} = 2^n + 2^n > 2^n + 2^{n_0-1}$ and $2^{n_- -1} = 2^{n-2} = 2^{n-1}\left(1-\frac{1}{2}\right) \le 2^{n-1}\left(1-\frac{1}{2^{n-n_0}}\right) = 2^{n-1} - 2^{n_0-1}$. If $n < -n_0$, then $-2^{|n_-|} = -2^{|n|+1} = -2^{|n|} - 2^{|n|} \le -2^{|n|} - 2^{n_0-1}$ and $-2^{|n_+|-1} = -2^{|n|-2} = -2^{|n|-1}\left(1-\frac{1}{2}\right) \ge -2^{|n|-1}\left(1-\frac{1}{2^{|n|-n_0}}\right) = -2^{|n|-1} + 2^{n_0-1}$. Finally, if $|n| \le n_0$, then $2^{n_+} = 2^{n_0+1} = 2^{n_0} + 2^{n_0} > 2^{|n|} + 2^{n_0-1}$ and $-2^{|n_-|} = -2^{n_0+1} = -2^{n_0} - 2^{n_0} < -2^{|n|} - 2^{n_0-1}$. Hence

$$\Lambda_n \subseteq \bigcup_{k=n_-}^{n_+} \gamma_k\,, \quad \forall n \in \mathbb{Z}. \tag{25}$$

Let

$$\begin{aligned}
\Omega_n &:= \left\{x \in \mathbb{R}^2 \;:\; 2^{n-1} < |x| < 2^n\right\},\; n \in \mathbb{Z},\\
\eta_n &:= \left\{x \in \mathbb{R}^2 \;:\; \frac{1}{L}2^{n-1} < |x| < L2^n\right\} \cap \Gamma,\\
\Gamma_n &:= \left\{x \in \mathbb{R}^2 \;:\; 2^{n-1} < |x| < 2^n\right\} \cap \Gamma.
\end{aligned}$$

Then there exist an integer n_1 such that $2^{n_1-1} \ge L$. One can take $n_1 = \lceil \log_2 L \rceil + 1$. This means that

$$\eta_n \subset \left\{x \in \mathbb{R}^2 \;:\; 2^{n-n_1} < |x| < 2^{n+n_1-1}\right\}$$

and

$$\eta_n \subseteq \bigcup_{k=n-n_1+1}^{n+n_1-1} \Gamma_k\,. \tag{26}$$

Hence, Lemma 2.2 implies

$$\|V\|^{(\mathrm{av})}_{\mathcal{B},\Gamma_n} \le \|V\|^{(\mathrm{av})}_{\mathcal{B},\eta_n} \le \beta \sum_{k=n-n_1+1}^{n+n_1-1} \|V\|^{(\mathrm{av})}_{\mathcal{B},\Gamma_k}\,, \tag{27}$$

where

$$\beta := \sup_{n\in\mathbb{Z}} \max_{k=n-n_1+1,\dots,n+n_1-1} \frac{|\eta_n|}{|\Gamma_k|}.$$

It is important to show that β is finite. For each $n - n_1 + 1 \le k \le n + n_1 - 1$, $|\Gamma_k| \ge 2 \cdot 2^{k-1} \ge 2^{n-n_1+1}$. Since η_n is a chord arc curve, there is a constant $K \ge 1$ such that

$$|\eta_n| \le 2LK2^n = 2^{n+1}KL.$$

Thus

$$\frac{|\eta_n|}{|\Gamma_k|} \le \frac{2^{n+1}KL}{2^{n-n_1+1}} = 2^{n_1}KL < \infty\,.$$

Let

$$Q_n := \int_{\gamma_n} |\log|x||V(x)\,ds(x), \quad n \in \mathbb{Z}\setminus\{0\}, \quad Q_0 := \int_{\gamma_0} V(x)\,ds(x),$$
$$R_n := \|V\|^{(\mathrm{av})}_{\mathcal{B},\Gamma_n}, \quad n \in \mathbb{Z}.$$

Theorem 4.3. There exist constants $c_1, c_2, C_3, C_4 > 0$ depending on L such that

$$N_-(q_V) \le 1 + C_3 \sum_{\{Q_n > c_1\}} \sqrt{Q_n} + C_4 \sum_{\{R_n > c_2\}} R_n, \qquad V \ge 0. \tag{28}$$

Proof. Let $\tilde{V}(y) := V(F(y))$, $u(y) := w(F(y))$, $w \in \mathrm{Dom}(q_V)$. Since F is bi-Lipschitz, there exist constants $C_5 > 0$ and $C_6 > 0$ depending on L such that

$$\begin{aligned}\int_{\mathbb{R}^2} |\nabla w(x)|^2\,dx &\ge \frac{1}{C_5}\int_{\mathbb{R}^2} |\nabla u(y)|^2\,dy\,,\\ \int_\Gamma V(x)|w(x)|^2\,ds(x) &\le C_6 \int_{\mathbb{R}} \tilde{V}(y_1,0)|u(y_1,0)|^2\,dy_1\,.\end{aligned}$$

Hence

$$N_-(q_V) \le N_-(q_{C_7\tilde{V}}), \tag{29}$$

where $C_7 := C_5C_6$ and

$$\begin{aligned} q_{C_7\tilde{V}}[u] &:= \int_{\mathbb{R}^2} |\nabla u(y)|^2\,dy - C_7\int_{\mathbb{R}} \tilde{V}(y_1,0)|u(y_1,0)|^2\,dy_1\,,\\ \mathrm{Dom}(q_{C_7\tilde{V}}) &= W^1_2(\mathbb{R}^2)\cap L^2(\mathbb{R}, \tilde{V}(.,0)\,dy_1).\end{aligned}$$

Let (r,θ) denote the polar coordinates in $\mathbb{R}^2$, $r \in \mathbb{R}_+,\ \theta \in [-\pi,\pi]$ and

$$u_{\mathcal{R}}(r) := \frac{1}{2\pi}\int_{-\pi}^{\pi} u(r,\theta)d\theta, \quad u_{\mathcal{N}}(r,\theta) := u(r,\theta) - u_{\mathcal{R}}(r), \quad u \in C(\mathbb{R}^2\setminus\{0\}).$$

Then

$$\int_{-\pi}^{\pi} u_{\mathcal{N}}(r,\theta)\,d\theta = 0, \quad \forall r > 0, \tag{30}$$

and it is easy to see that

$$\int_{\mathbb{R}^2} u_{\mathcal{R}} v_{\mathcal{N}}\,dy = 0, \quad \forall u, v \in C\left(\mathbb{R}^2\setminus\{0\}\right)\cap L^2\left(\mathbb{R}^2\right).$$

Using the representation of the gradient in polar coordinates one can easily show that

$$\int_{\mathbb{R}^2} \nabla u_{\mathcal{R}} \nabla v_{\mathcal{N}}\,dy = 0, \quad \forall u, v \in C_0^\infty\left(\mathbb{R}^2\right).$$

Hence $u \mapsto Pu := u_{\mathcal{R}}$ extends to an orthogonal projection $P : W_2^1(\mathbb{R}^2) \to W_2^1(\mathbb{R}^2)$. Since

$$\int_{\mathbb{R}^2} |\nabla u|^2\, dy = \int_{\mathbb{R}^2} |\nabla u_{\mathcal{R}}|^2\, dy + \int_{\mathbb{R}^2} |\nabla u_{\mathcal{N}}|^2\, dy,$$
$$\int_{\mathbb{R}} \tilde{V}|u|^2\, dy_1 \le 2\int_{\mathbb{R}} \tilde{V}|u_{\mathcal{R}}|^2\, dy_1 + 2\int_{\mathbb{R}} \tilde{V}|u_{\mathcal{N}}|^2\, dy_1,$$

one has

$$N_-(q_{C_7\tilde{V}}) \le N_-(q_{1,2C_7\tilde{V}}) + N_-(q_{2,2C_7\tilde{V}}) \tag{31}$$

where $N_-(q_{1,2C_7\tilde{V}})$ and $N_-(q_{2,2C_7\tilde{V}})$ are the restrictions of the form $N_-(q_{C_7\tilde{V}})$ to $PW_2^1(\mathbb{R}^2)$ and $(I-P)W_2^1(\mathbb{R}^2)$ respectively.

To estimate the right-hand side of (31), we start with the case of

$$PW_2^1(\mathbb{R}^2) = \{u \in W_2^1(\mathbb{R}^2) \ :\ u(x) = u_{\mathcal{R}}(r)\}.$$

Let $\tilde{V}_*(r) := \tilde{V}(r) + \tilde{V}(-r)$. Then

$$\int_{\mathbb{R}} \tilde{V}(y_1, 0)|u_{\mathcal{R}}(y_1, 0)|^2\, dy_1 = \int_{\mathbb{R}_+} \tilde{V}_*(r)|u_{\mathcal{R}}(r)|^2\, dr$$

Using the exponential change of variables $r = e^t$ and using the notation $u_{\mathcal{R}}(y) = u_{\mathcal{R}}(r) = v(t)$, we have

$$\int_{\mathbb{R}^2} |\nabla u_{\mathcal{R}}(y)|^2\, dy = 2\pi \int_{\mathbb{R}} |v'(t)|^2\, dt$$

and

$$C_7 \int_{\mathbb{R}_+} \tilde{V}_*(r)|u_{\mathcal{R}}(r)|^2\, dr = 2\pi \int_{\mathbb{R}} G(t)|v(t)|^2\, dt,$$

where

$$G(t) := \frac{e^t}{2\pi} C_7 \tilde{V}_*(e^t). \tag{32}$$

Hence, we have the following well studied one-dimensional Schrödinger operator on $L^2(\mathbb{R})$

$$-\frac{d^2}{dt^2} - 2G, \quad G \ge 0.$$

Let

$$\begin{aligned} q_{2G}[v] &:= \int_{\mathbb{R}} |v'(t)|^2\, dt - 2\int_{\mathbb{R}} G(t)|v(t)|^2\, dt, \\ \text{Dom } (q_{2G}) &= W_2^1(\mathbb{R}) \cap L^2(\mathbb{R}, Gdt). \end{aligned}$$

Furthermore, let

$$I_n := [2^{n-1}, 2^n],\ n > 0, \quad I_0 := [-1, 1], \quad I_n := [-2^{|n|}, -2^{|n|-1}],\ n < 0$$

and

$$\mathcal{Q}_n := \int_{I_n} |t| G(t)\, dt,\ \ n \neq 0, \quad \mathcal{Q}_0 := \int_{I_0} G(t)\, dt. \tag{33}$$

Then one has

$$N_-(q_{1,2C_7\tilde{V}}) \le N_-(q_{2G}) \le 1 + 7.61 \sum_{\{\mathcal{Q}_n > 0.046\}} \sqrt{\mathcal{Q}_n} \tag{34}$$

(see the estimate before (39) in [18]). Next, we consider the case of

$$(I - P)W_2^1(\mathbb{R}^2) = \{u \in W_2^1(\mathbb{R}^2)\ :\ u(y) = u_{\mathcal{N}}(y)\}$$

to obtain an estimate for the second term in (31). Let $J_n := \Omega_n \cap \mathbb{R}$, $n \in \mathbb{Z}$ and

$$\begin{aligned}
& q_{2,2C_7\tilde{V}}[u_{\mathcal{N}}] := \int_{\mathbb{R}^2} |\nabla u_{\mathcal{N}}(y)|^2\, dy - 2C_7 \int_{\mathbb{R}} \tilde{V}(y_1, 0)|u_{\mathcal{N}}(y_1, 0)|^2\, dy_1, \\
& \mathrm{Dom}(q_{2,2C_7\tilde{V}}) = (I - P)W_2^1(\mathbb{R}^2) \cap L^2(\tilde{V}(.,0)\, dy_1), \\
& q_{2,2C_7\tilde{V},\Omega_n}[u] := \int_{\Omega_n} |\nabla u(y)|^2\, dy - 2C_7 \int_{J_n} \tilde{V}(y_1)|u(y_1)|^2\, dy_1, \\
& \mathrm{Dom}\,(q_{2,2C_7\tilde{V},\Omega_n}) = \left\{u \in W_2^1(\Omega_n) \cap L^2\left(J_n, \tilde{V} dy_1\right)\ :\ u_{\Omega_n} = 0\right\},
\end{aligned}$$

where $u_Q := \frac{1}{|Q|}\int_Q u(y)\, dy$. Note that $v_{\Omega_n} = 0$ for any $v \in (I - P)W_2^1(\mathbb{R}^2)$. Using [17, Lemma 3] instead of [18, Lemma 7.6] in the proof of [18, Lemma 7.7], one can show similarly to [18, Lemma 7.8] that there is a constant $C_8 > 0$ such that

$$N_-(q_{2,2C_7\tilde{V},\Omega_n}) \le C_8 \|\tilde{V}\|^{(\mathrm{av})}_{\mathcal{B},J_n}, \quad \forall\, \tilde{V} \ge 0.$$

Let $D_n := \|\tilde{V}\|^{(\mathrm{av})}_{\mathcal{B},J_n}$. Then for any $c < \frac{1}{C_8}$, the variational principle (see, e.g., [10, Ch.6, § 2.1, Theorem 4]) implies

$$N_-(q_{2,2C_7\tilde{V}}) \le C_8 \sum_{\{D_n > c\,:\, n\in\mathbb{Z}\}} D_n, \quad \forall\, \tilde{V} \ge 0. \tag{35}$$

If $\|\tilde{V}\|^{(\mathrm{av})}_{\mathcal{B},J_n} < \frac{1}{C_8}$, then $N_-(q_{2,2C_7\tilde{V},J_n}) = 0$ and one can drop this term from the sum in (35). Now it follows from (31), (34) and (35) that

$$N_-(q_{C_7\tilde{V}}) \le 1 + 7.61 \sum_{\{\mathcal{Q}_n > 0.046\}} \sqrt{\mathcal{Q}_n} + C_8 \sum_{\{D_n > c\,:\, n\in\mathbb{Z}\}} D_n, \ \ \forall\, \tilde{V} \ge 0. \tag{36}$$

It now remains to write the estimate for $N_-(q_{C_7\tilde{V}})$ in terms of the original potential V to obtain an estimate for $N_-(q_V)$ (see (28)).
Let $\jmath_n := F(\{y \in \mathbb{R} : |y| \in U_n\})$. Then we have for $n \neq 0$ by (32) and (33)

$$\begin{aligned}
\mathcal{Q}_n &= \int_{I_n} |t| G(t)\, dt = \frac{1}{2\pi} C_7 \int_{I_n} |t| e^t \tilde{V}_*(e^t)\, dt \\
&= \frac{1}{2\pi} C_7 \int_{U_n} |\log r| \tilde{V}_*(r)\, dr \\
&= \frac{1}{2\pi} C_7 \int_{U_n} |\log y_1| \tilde{V}_*(y_1)\, dy_1 \\
&\leq \frac{1}{2\pi} C_7 \left(L \log L \int_{\jmath_n} V(x)\, ds(x) + L \int_{\jmath_n} |\log |x|| V(x)\, ds(x) \right) \\
&\leq \frac{1}{2\pi} C_7 L \max\{1, \log L\} \left(\int_{\jmath_n} V(x)\, ds(x) + \int_{\jmath_n} |\log |x|| V(x)\, ds(x) \right) \\
&\leq \frac{1}{2\pi} C_7 L \max\{1, \log L\} \left(\int_{\Lambda_n} V(x)\, ds(x) + \int_{\Lambda_n} |\log |x|| V(x)\, ds(x) \right) \\
&\leq \frac{1}{2\pi} C_7 L \max\{1, \log L\} \sum_{k=n_-}^{n_+} \left(\int_{\gamma_k} V(x)\, ds(x) + \int_{\gamma_k} |\log |x|| V(x)\, ds(x) \right) \\
&\leq C_9 \sum_{k=n_-}^{n_+} Q_k,
\end{aligned}$$

where $C_9 := \frac{1}{2\pi} C_7 L \max\{1, \log L\}$ (see (25)). For $n = 0$,

$$\begin{aligned}
\mathcal{Q}_0 &= \int_{I_0} G(t)\, dt = \frac{1}{2\pi} C_7 \int_{I_0} e^t \tilde{V}_*(e^t)\, dt = \frac{1}{2\pi} C_7 \int_{U_0} \tilde{V}_*(r)\, dr \\
&= \frac{1}{2\pi} C_7 \int_{U_0} \tilde{V}_*(y_1)\, dy_1 \leq \frac{1}{2\pi} C_7 L \int_{\jmath_0} V(x) ds(x) \\
&\leq \frac{1}{2\pi} C_7 L \int_{\Lambda_0} V(x)\, ds(x) \leq \frac{1}{2\pi} C_7 L \sum_{k=-n_0-1}^{n_0+1} \int_{\gamma_k} V(x)\, ds(x) \\
&\leq C_9 \sum_{k=-n_0-1}^{n_0+1} Q_k.
\end{aligned}$$

So,

$$\mathcal{Q}_n \leq C_9 \sum_{k=n_-}^{n_+} Q_k, \quad \forall n \in \mathbb{Z}. \tag{37}$$

For each $n \in \mathbb{Z}$, there exists $n^* \in [n_-, n_+] \cap \mathbb{Z}$ such that

$$Q_{n^*} = \max_{k=n_-,\dots,n_+} Q_k.$$

Note that for any $m \in \mathbb{Z}$, there are no more than $2n_0 + 5$ numbers $n \in \mathbb{Z}$ such that $m \in [n_-, n_+]$. Hence

$$7.61 \sum_{\{\mathcal{Q}_n > 0.046\}} \sqrt{\mathcal{Q}_n} \;\le\; 7.61 \sum_{\left\{Q_{n^*} > \frac{0.046}{C_9(2n_0+3)}\right\}} \sqrt{C_9(2n_0+3)Q_{n^*}}$$
$$\le\; C_3 \sum_{\{Q_n > c_1\}} \sqrt{Q_n}\,, \qquad (38)$$

where $C_3 := 7.61(2n_0+5)\sqrt{C_9(2n_0+3)}\,,\;\; c_1 := \frac{0.046}{C_9(2n_0+3)}$.
Let $\ell_n := F(J_n)$. Since $\tilde{V}(y_1) = V(F(y_1))$, we have

$$\begin{aligned} D_n = \|\tilde{V}\|^{(\mathrm{av})}_{\mathcal{B},J_n} &= \sup\left\{\left|\int_{J_n} \tilde{V}(y_1)g(y_1)\,dy_1\right| : \int_{J_n} \mathcal{A}(|g(y_1)|)dy_1 \le |J_n|\right\} \\ &= \sup\left\{\left|\int_{J_n} V(F(y_1))g(y_1)\,dy_1\right| : \int_{J_n} \mathcal{A}(|g(y_1)|)dy_1 \le |J_n|\right\} \\ &\le \sup\left\{L\left|\int_{\ell_n} V(x)h(x)\,ds(x)\right| : \int_{\ell_n} \mathcal{A}(|h(x)|)ds(x) \le L|\ell_n|\right\} \\ &\le L\sup\left\{\left|\int_{\eta_n} V(x)h(x)\,ds(x)\right| : \int_{\eta_n} \mathcal{A}(|h(x)|)ds(x) \le L|\eta_n|\right\} \end{aligned}$$
$$= L\|V\|^{(\mathrm{av}),L}_{\mathcal{B},\eta_n} \le L^2\|V\|^{(\mathrm{av})}_{\mathcal{B},\eta_n} \le L^2\beta \sum_{k=n-n_1+1}^{n+n_1-1} \|V\|^{(\mathrm{av})}_{\mathcal{B},\Gamma_k} = L^2\beta \sum_{k=n-n_1+1}^{n+n_1-1} R_k$$

(see (12) and (27)).
For each $n \in \mathbb{Z}$, there exists $n^\dagger \in [n - n_1 + 1, n + n_1 - 1] \cap \mathbb{Z}$ such that

$$R_{n^\dagger} = \max_{k=n-n_1+1,\dots,n+n_1-1} R_k.$$

Note that for any $m \in \mathbb{Z}$, there are $2n_1 - 1$ numbers $n \in \mathbb{Z}$ such that $m \in [n - n_1 + 1, n + n_1 - 1]$. Hence

$$C_8 \sum_{\{D_n > c\,:\, n\in\mathbb{Z}\}} D_n \le C_8 \sum_{\left\{R_{n^\dagger} > \frac{c}{L^2\beta\,(2n_1-1)}\right\}} L^2\beta\,(2n_1-1)R_{n^\dagger} \le C_4 \sum_{R_n > c_2} R_n, \quad (39)$$

where $C_4 := C_8L^2\beta\,(2n_1-1)^2$ and $c_2 := \frac{c}{L^2\beta\,(2n_1-1)}$. This together with (36) and (38) imply (28). □

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