On the discrete spectrum of Schrödinger operators with Ahlfors regular potentials in a strip

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Abstract

In this paper, quantitative upper estimates for the number of eigenvalues lying below the essential spectrum of Schrödinger operators with potentials generated by Ahlfors regular measures in a strip subject to two different types of boundary conditions (Robin and Dirichlet respectively) are presented. The estimates are presented in terms of weighted L^1 norms and Orlicz norms of the potential.

Keywords: Discrete spectrum; Schrödinger operators; Ahlfors regular potential; strip.
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1 Introduction

Let $V \in L^1_{\text{loc}}(\mathbb{R}^d)$. According to the Cwikel-Lieb-Rozenblum (CLR) inequality (see, e.g., [3, 25]), the number of negative eigenvalues of the Schrödinger operator $-\Delta - V, V \geq 0$ on $L^2(\mathbb{R}^d)$ with $d \geq 3$ is estimated above by $\|V\|_{L^{\frac{d}{2}}(\mathbb{R}^d)}$. In the case $d = 2$, this estimate fails, but there has been significant recent progress in obtaining estimates of the CLR-type in two-dimensions and the best known estimates have been obtained in [26]. Estimates for the number of negative eigenvalues of Schrödinger operators with potential of the form $V\mu$, where μ is a Radon measure and V is an appropriate function, were obtained in [6], and results from [26] were extended to this setting in [13, 16]. In the present paper, we obtain estimates for the number of eigenvalues below the essential spectrum of a Schrödinger operator with

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potential of the form $V\mu$ similar to those in [16] in a strip subject to boundary conditions of the Robin type. Similar estimates are also obtained when the domain of the operator is characterized by Dirichlet boundary conditions (see remark 6.4). Below is a precise description of the operator studied herein.

Let $S := \mathbb{R} \times (0, a)$, $a > 0$ be a strip, μ a σ-finite positive Radon measure on $\mathbb{R}^2$ and $V : \mathbb{R}^2 \longrightarrow \mathbb{R}$ a non-negative function integrable on bounded subsets of S with respect to μ. We study the following Schrödinger operator

$$H_\mu := -\Delta - V\mu\,, \quad V \geq 0, \qquad \text{on} \;\; L^2(S), \tag{1}$$

where $\Delta := \sum_{k=1}^{2} \frac{\partial^2}{\partial x_k^2}$, subject to the following Robin boundary conditions

$$u_{x_2}(x_1, 0) + \alpha u(x_1, 0) = u_{x_2}(x_1, a) + \beta u(x_1, a) = 0, \tag{2}$$

where $\alpha, \beta \in \mathbb{R}$. Here $u_{x_i} (i = 1, 2)$ denotes the partial derivative of u with respect to x_i and note that μ does not have to be the two-dimensional Lebesgue measure. A physical motivation to this problem is closely related to the study of spectral properties of quantum waveguides (see, e.g., [8, 9, 18, 21, 24]).

Under certain assumptions about V and μ, (1) is well defined and self-adjoint on $L^2(S)$ and its essential spectrum is the interval $[\lambda_1, +\infty)$, where λ_1 is the first eigenvalue of $-\Delta = -\partial^2_{x_2}$ considered along the width of the strip with boundary conditions (2) (see, e.g., [2, 12, 18, 29]). For a detailed discussion about what λ_1 is, see e.g. [14]. The case where both α and β are equal to zero with μ being the two-dimensional Lebesgue measure has been previously studied by A. Grigor'yan and N. Nadirashvili [11] who obtained estimates in terms of weighted L^1 norms and $L^p, p > 1$ norms of V. Below, we prove stronger results in our more general setting.

Let $\mathcal{H}$ be a Hilbert space and let $\mathbf{q}$ be a Hermitian form with a domain $\mathrm{Dom}\,(\mathbf{q}) \subseteq \mathcal{H}$. Set

$$N_-(\mathbf{q}) := \sup\left\{\dim \mathcal{L} \mid \mathbf{q}[u] < 0,\, \forall u \in \mathcal{L} \setminus \{0\}\right\}, \tag{3}$$

where $\mathcal{L}$ denotes a linear subspace of $\mathrm{Dom}\,(\mathbf{q})$. The number $N_-(\mathbf{q})$ is called the Morse index of $\mathbf{q}$ in $\mathrm{Dom}\,(\mathbf{q})$. If $\mathbf{q}$ is the quadratic form of a self-adjoint operator A with no essential spectrum in $(-\infty, 0)$, then $N_-(\mathbf{q})$ is the number of negative eigenvalues of A repeated according to their multiplicity (see, e.g., [4, S1.3] or [5, Theorem 10.2.3]).
Estimating the number of eigenvalues of (1) below its essential spectrum is equivalent to estimating the number of negative eigenvalues of the operator

$$H_{\lambda_1,\mu} = -\Delta - \lambda_1 - V\mu \text{ on } L^2(S) \tag{4}$$

subject to boundary conditions in (2). Now, defining (4) via its quadratic form we have

$$\begin{aligned}\mathcal{E}_{\lambda_1,\mu,S}[u] &:= \int_S |\nabla u(x)|^2\,dx - \lambda_1 \int_S |u(x)|^2\,dx - \alpha \int_{\mathbb{R}} |u(x_1,0)|^2\,dx_1 \\ &+ \beta \int_{\mathbb{R}} |u(x_1,a)|^2\,dx_1 - \int_{\overline{S}} V(x)|u(x)|^2\,d\mu(x), \qquad (5)\\ \mathrm{Dom}\,(\mathcal{E}_{\lambda_1,\mu,S}) &= \left\{u \in W_2^1(S) \cap L^2(\overline{S}, V d\mu\right\}.\end{aligned}$$

Note that we take the closure $\overline{S}$ of the open strip S in the terms involving μ as this measure might charge subsets of the horizontal lines $x_2 = 0$ and $x_2 = a$.
We denote by $N_-(\mathcal{E}_{\lambda_1,\mu,S})$ the number of negative eigenvalues of (4) counting multiplicities.

Let $S_n := (n, n+1) \times (0, a)$, $n \in \mathbb{Z}$. Then it follows from the variational principle [13, Lemma 1.6.2] (see also [7, Ch.6, §2.1, Theorem 4] for the case when μ is absolutely continuous with respect to the Lebesgue measure) that

$$N_-(\mathcal{E}_{\lambda_1,\mu,S}) \le \sum_{n\in\mathbb{Z}} N_-(\mathcal{E}_{\lambda_1,\mu,S_n}), \qquad (6)$$

where $N_-(\mathcal{E}_{\lambda_1,\mu,S_n})$ are the restrictions of the form $\mathcal{E}_{\lambda_1,\mu,S}$ to $\overline{S_n}$. Let $u(x) = u_1(x_2)$, where u_1 is an eigenfunction of $-\Delta = -\partial^2_{x_2}$ considered along the width of the strip with boundary conditions (2) corresponding to the first eigenvalue λ_1. Then it is easy to see that

$$\mathcal{E}_{\lambda_1,\mu,S_n}[u] = -\int_{\overline{S_n}} V(x)|u_1(x_2)|^2\,d\mu(x)$$

and since $V \ge 0$, the right-hand side is strictly negative unless

$$\mu\Big(\mathrm{supp}\big(V|u_1|^2\big) \cap \overline{S_n}\Big) = 0.$$

So, one usually has $N_-(\mathcal{E}_{\lambda_1,\mu,S_n}) \ge 1$ and thus the right-hand side of (6) diverges. To avoid this, we shall split the problem into two problems. The first will be defined by the restriction of the form to the subspace of functions obtained by multiplying $u_1(x_2)$ by functions depending only on x_1, and is thus reduced to a well studied one-dimensional Schrödinger operator. The second problem will be defined by a class of functions orthogonal to u_1 in the $L^2((0,a))$ inner product.

2 Notation

In order to state the estimate for $N_-(\mathcal{E}_{\lambda_1,\mu,S})$, we need some notation from the theory of Orlicz spaces (see, e.g., [1, 17, 22]). Let Φ and Ψ be mutually complementary N-functions, and let $L_\Phi(\Omega,\mu)$, $L_\Psi(\Omega,\mu)$ be the corresponding Orlicz spaces. We will use the following norms on $L_\Psi(\Omega,\mu)$

$$\|f\|_{\Psi,\mu} = \|f\|_{\Psi,\Omega,\mu} = \sup\left\{\left|\int_\Omega fg d\mu\right| : \int_\Omega \Phi(|g|)d\mu \le 1\right\} \tag{7}$$

and

$$\|f\|_{(\Psi,\mu)} = \|f\|_{(\Psi,\Omega,\mu)} = \inf\left\{\kappa > 0 : \int_\Omega \Psi\left(\frac{|f|}{\kappa}\right) d\mu \le 1\right\}. \tag{8}$$

These two norms are equivalent

$$\|f\|_{(\Psi,\mu)} \le \|f\|_{\Psi,\mu} \le 2\|f\|_{(\Psi,\mu)}\,, \quad \forall f \in L_\Psi(\Omega), \tag{9}$$

(see [1]).
Note that

$$\int_\Omega \Psi\left(\frac{|f|}{\kappa_0}\right) d\mu \le C_0, \quad C_0 \ge 1 \implies \|f\|_{(\Psi)} \le C_0\kappa_0. \tag{10}$$

Indeed, since Ψ is convex and increasing on $[0,+\infty)$, and $\Psi(0) = 0$, we get for any $\kappa \ge C_0\kappa_0$,

$$\int_\Omega \Psi\left(\frac{|f|}{\kappa}\right) d\mu \le \int_\Omega \Psi\left(\frac{|f|}{C_0\kappa_0}\right) d\mu \le \frac{1}{C_0}\int_\Omega \Psi\left(\frac{|f|}{\kappa_0}\right) d\mu \le 1 \tag{11}$$

(see [26]). It follows from (10) with $\kappa_0 = 1$ that

$$\|f\|_{(\Psi,\mu)} \le \max\left\{1, \int_\Omega \Psi(|f|)d\mu\right\}. \tag{12}$$

We will also need the following equivalent norm on $L_\Psi(\Omega)$ with $\mu(\Omega) < \infty$, which was introduced in [27]:

$$\|f\|^{(av)}_{\Psi,\Omega} := \sup\left\{\left|\int_\Omega fg\, d\mu\right| : \int_\Omega \Phi(|g|)d\mu \le \mu(\Omega)\right\}. \tag{13}$$

We will use the following pair of mutually complementary N-functions

$$\mathcal{A}(s) = e^{|s|} - 1 - |s|, \quad \mathcal{B}(s) = (1 + |s|)\ln(1 + |s|) - |s|, \quad s \in \mathbb{R}.$$

Definition 2.1. *Let μ be a positive Radon measure on $\mathbb{R}^2$. We say the measure μ is Ahlfors regular of dimension $d > 0$ if there exist positive constants c_0 and c_1 such that*

$$c_0 r^d \le \mu(B(x,r)) \le c_1 r^d \tag{14}$$

for all $0 < r \le diam(supp\,\mu)$ and all $x \in supp\,\mu$, where $B(x,r)$ is the ball of radius r centred at x, and the constants c_0 and c_1 are independent of the balls.

Definition 2.2. (Local Ahlfors regularity) *We say that a measure μ is locally Ahlfors regular on a bounded set $G \subset \mathbb{R}^2$ if for every $R < \infty$ there exist $d > 0$ and positive constants $c_0(R)$ and $c_1(R)$ such that*

$$c_0(R)\, r^d \le \mu(B(x,r)) \le c_1(R)\, r^d \tag{15}$$

for all $0 < r \le R$ and all $x \in supp\,\mu \cap \overline{G}$. We say that μ is locally Ahlfors regular on the strip S if (15) *holds for all $0 < r \le R$ and all $x \in supp\,\mu \cap \overline{S}$, and there exist constants $c_2, c_3 > 0$ such that*

$$c_2 \mu\left(\overline{S_{n\pm1}}\right) \le \mu\left(\overline{S_n}\right) \le c_3 \mu\left(\overline{S_{n\pm1}}\right), \quad \forall n \in \mathbb{Z}\,. \tag{16}$$

Thus for each $n \in \mathbb{Z}$,

$$c_2^k \mu\left(\overline{S_{n\pm k}}\right) \le \mu\left(\overline{S_n}\right) \le c_3^k \mu\left(\overline{S_{n\pm k}}\right), \quad \forall k \in \mathbb{N}\,. \tag{17}$$

From now onwards, it will be assumed that μ is a σ-finite positive Radon measure that is locally Ahlfors regular on S.

3 Statement of the main result

Let

$$I_n := [2^{n-1}, 2^n],\ n > 0, \quad I_0 := [-1,1], \quad I_n := [-2^{|n|}, -2^{|n|-1}],\ n < 0,$$

$$\begin{aligned}
\mathcal{F}_n &:= \int_{I_n}\int_0^a |x_1| V(x) |u_1(x_2)|^2 \, d\mu(x) \quad n \ne 0\,,\\
\mathcal{F}_0 &:= \int_{I_0}\int_0^a V(x) |u_1(x_2)|^2 \, d\mu(x)\,,\\
M_n &:= \|V\|_{\mathcal{B},\overline{S_n},\mu}\,,
\end{aligned}$$

where u_1 is a normalized eigenfunction of $-\Delta = -\partial_{x_2}^2$ on $(0,a)$ with boundary conditions (2) corresponding to the first eigenvalue λ_1 (here the normalization is with respect to the Lebesgue measure).

Theorem 3.1. *Let μ be a σ-finite positive Radon measure on $\mathbb{R}^2$ that is locally Ahlfors regular on S and $V \in L_{\mathcal{B}}(S_n, \mu)$ for every $n \in \mathbb{Z}$. Then there exist constants $C, c > 0$ such that*

$$N_-\left(\mathcal{E}_{\lambda_1,\mu,S}\right) \le 1 + C\left(\sum_{\{\mathcal{F}_n > c,\, n\in\mathbb{Z}\}} \sqrt{\mathcal{F}_n} + \sum_{\{M_n > c,\, n\in\mathbb{Z}\}} M_n\right). \tag{18}$$

4 Auxiliary results

We start with a result that was obtained in [16] (see also [13, Lemma 3.1.1]). For the reader's convenience, we reproduce the proof here.

Lemma 4.1. *Let μ be a σ-finite Radon measure on $\mathbb{R}^2$ such that $\mu(\{x\}) = 0$ for all $x \in \mathbb{R}^2$. Let*

$$\Sigma := \{\theta \in [0, \pi) \ : \ \exists\, l_\theta \text{ such that } \mu(l_\theta) > 0\}, \tag{19}$$

where l_θ is a line in $\mathbb{R}^2$ in the direction of the vector $(\cos\theta, \sin\theta)$. Then Σ is at most countable.

Proof. Let

$$\Sigma_N := \{\theta \in [0, \pi) \ : \ \exists\, l_\theta \text{ such that } \mu(l_\theta \cap B(0, N)) > 0\},$$

where $B(0, N)$ is the ball of radius $N \in \mathbb{N}$ centred at 0. Then

$$\Sigma = \bigcup_{N\in\mathbb{N}} \Sigma_N.$$

It is now enough to show that Σ_N is at most countable for $\forall N \in \mathbb{N}$. Suppose that Σ_N is uncountable. Then there exists a $\delta > 0$ such that

$$\Sigma_{N,\delta} := \{\theta \in [0, \pi) \ : \ \exists\, l_\theta \text{ such that } \mu(l_\theta \cap B(0, N)) > \delta\}$$

is infinite. Otherwise, $\Sigma_N = \bigcup_{n\in\mathbb{N}} \Sigma_{N,\frac{1}{n}}$ would have been finite or countable. Now take $\theta_1, ..., \theta_k, ... \in \Sigma_{N,\delta}$. Then

$$\mu\left(l_{\theta_k} \cap B(0, N)\right) > \delta, \quad \forall k \in \mathbb{N}.$$

Since $l_{\theta_j} \cap l_{\theta_k}$, $j \ne k$ contains at most one point, then

$$\mu\left(\bigcup_{j\ne k} (l_{\theta_j} \cap l_{\theta_k})\right) = 0.$$

Let

$$\tilde{l}_{\theta_k} := l_{\theta_k} \setminus \underset{j\neq k}{\cup} (l_{\theta_j} \cap l_{\theta_k})\,.$$

Then $\tilde{l}_{\theta_j} \cap \tilde{l}_{\theta_k} = \emptyset,\ j \neq k$ and $\tilde{l}_{\theta_k} \cap B(0,N) \subset B(0,N)$. So

$$\mu\left(\underset{k\in\mathbb{N}}{\cup}(\tilde{l}_{\theta_k} \cap B(0,N))\right) \leq \mu\left(B(0,N)\right) < \infty\,.$$

But

$$\mu\left(\tilde{l}_{\theta_k} \cap B(0,N)\right) = \mu\left(l_{\theta_k} \cap B(0,N)\right) \geq \delta$$

which implies

$$\sum_{k\in\mathbb{N}} \mu\left(\tilde{l}_{\theta_k} \cap B(0,N)\right) \geq \sum_{k\in\mathbb{N}} \delta = \infty\,.$$

This contradiction means that Σ_N is at most countable for each $N \in \mathbb{N}$. Hence Σ is at most countable. $\square$

Corollary 4.2. *There exists* $\theta_0 \in [0,\pi)$ *such that* $\theta_0 \notin \Sigma$ *and* $\theta_0 + \frac{\pi}{2} \notin \Sigma$.

Proof. The set

$$\Sigma - \frac{\pi}{2} := \left\{\theta - \frac{\pi}{2}\ : \theta \in \Sigma\right\}$$

is at most countable. This implies that there exists a $\theta_0 \notin \Sigma \cup (\Sigma - \frac{\pi}{2})$. Thus $\theta_0 + \frac{\pi}{2} \notin \Sigma$. $\square$

Let $G \subset \mathbb{R}^2$ be a bounded set with Lipschitz boundary such that $0 < \mu(\overline{G}) < \infty$. Let G_* be the smallest square containing G with sides chosen in the directions θ_0 and $\theta_0 + \frac{\pi}{2}$ from Corollary 4.2, and let G^* be the closed square with the same centre as G_* and sides in the same direction but of length 3 times that of G_*. Let

$$\kappa_0(G) := \frac{\mu(G^*)}{\mu\left(\overline{G}\right)}\,.$$

There exists a bounded linear operator

$$T : W_2^1(G) \longrightarrow W_2^1(\mathbb{R}^2)$$

which satisfies

$$Tu|_G = u, \quad \forall u \in W_2^1(G)$$

(see, e.g., [28, Ch.VI, Theorem 5]). We will use the following notation:

$$u_E := \frac{1}{|E|}\int_E u(x)\,dx,$$

where $E \subseteq \mathbb{R}^2$ is a set of a finite two dimensional Lebesgue measure $|E|$. The following result is similar to [13, Lemma 3.2.13] and follows directly from the proof of the latter.

Lemma 4.3. *Let G be as above and μ be a σ-finite positive Radon measure on $\mathbb{R}^2$ that is locally Ahlfors regular on G. Choose and fix a direction satisfying Corollary 4.2. Further, for all $x \in \overline{G}$ and for all $r > 0$, let $\Delta_x(r)$ be a square with edges of length r in the chosen direction centred at $x \in supp\,\mu \cap \overline{G}$. Then for any $V \in L_{\mathcal{B}}\left(\overline{G}, \mu\right)$, $V \geq 0$ and any $m \in \mathbb{N}$ there exists a finite cover of $supp\,\mu \cap \overline{G}$ by squares $Q_{x_k}(r_{x_k}), r_{x_k} > 0, k = 1, 2, ..., m_0$, such that $m_0 \leq m$ and*

$$\int_{\overline{G}} V(x)|u(x)|^2 d\mu(x) \leq C(G)C(d)\frac{c_1(R)}{c_0(R)}\kappa_0^2 m^{-1}\|V\|^{(av)}_{\mathcal{B},\overline{G},\mu}\|u\|^2_{W^1_2(G)} \qquad (20)$$

for all $u \in W^1_2(G) \cap C(\overline{G})$ with $(Tu)_{Q_{x_k}(r_{x_k})} = 0, k = 1, ..., m_0$, where the constant $C(G)$ depends only on G and is invariant under parallel translations of G, $C(d)$ depends only on d in (15), and R is the diameter of G^*. If $m = 1$, one can take $m_0 = 0$.

Let

$$\mathcal{E}_{S_n}[u] := \int_{S_n} |\nabla u(x)|^2\, dx + \beta \int_n^{n+1} |u(x_1, a)|^2\, dx_1 - \alpha \int_n^{n+1} |u(x_1, 0)|^2\, dx_1, \qquad (21)$$

for all $u \in W^1_2(S_n)$.
Let $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_j \leq \cdots$ be the eigenvalues of the above boundary value problem. By the Min-Max principle, we have

$$\lambda_1 = \min_{\substack{u \in W^1_2(S_n)\\ u \neq 0}} \frac{\mathcal{E}_{S_n}[u]}{\int_{S_n} |u(x)|^2\, dx},$$

$$\lambda_2 = \min_{\substack{u \in W^1_2(S_n)\\ u \neq 0, u \perp u_1}} \frac{\mathcal{E}_{S_n}[u]}{\int_{S_n} |u(x)|^2\, dx},$$

where u_1 is a normalized eigenfunction corresponding to λ_1. The function u_1 does not depend on x_1 and, viewed as a function of one variable x_2, it is a normalized eigenfunction of $-\Delta = -\partial^2_{x_2}$ on $(0, a)$ with boundary conditions (2) corresponding to the first eigenvalue λ_1, moreover $\lambda_1 < \lambda_2$ (see [14] or [13, Section 1.5]).
It follows from the above that for all $u \in W^1_2(S_n)$, $u \perp u_1$, one has

$$\lambda_2 \int_{S_n} |u(x)|^2\, dx \leq \mathcal{E}_{S_n}[u],$$

which in turn implies

$$\begin{aligned}\mathcal{E}_{S_n}[u] - \lambda_1 \int_{S_n} \mid u(x) \mid^2 \, dx &= \mathcal{E}_{S_n}[u] - \lambda_2 \int_{S_n} \mid u(x) \mid^2 \, dx \\ &+ (\lambda_2 - \lambda_1) \int_{S_n} |u(x)|^2 \, dx \\ &\geq (\lambda_2 - \lambda_1) \int_{S_n} |u(x)|^2 \, dx.\end{aligned}$$

Since $\lambda_1 < \lambda_2$, one gets

$$\int_{S_n} |u(x)|^2 \, dx \leq \frac{1}{\lambda_2 - \lambda_1} \left(\mathcal{E}_{S_n}[u] - \lambda_1 \int_{S_n} |u(x)|^2 \, dx \right), \quad \forall u \in W_2^1(S_n), \;\; u \perp u_1. \tag{22}$$

Lemma 4.4. [Ehrling's Lemma] *Let X_0, X_1 and X_2 be Banach spaces such that $X_2 \hookrightarrow X_1$ is compact and $X_1 \hookrightarrow X_0$. Then for every $\varepsilon > 0$, there exists a constant $C(\varepsilon) > 0$ such that*

$$\|u\|_{X_1} \leq \varepsilon \|u\|_{X_2} + C(\varepsilon) \|u\|_{X_0}, \qquad \forall u \in X_2\,. \tag{23}$$

See, e.g., [23] for details and proof.
Let $S(\mathbb{R}^2)$ be the class of all functions $\varphi \in C^\infty(\mathbb{R}^2)$ such that for any multi-index γ and any $k \in \mathbb{N}$,

$$\sup_{x \in \mathbb{R}^2} (1 + |x|)^k |\partial^\gamma \varphi(x)| < \infty.$$

Denote by $S'(\mathbb{R}^2)$ the dual space of $S(\mathbb{R}^2)$. For $s > 0$, let

$$H^s(\mathbb{R}^2) := \left\{ u \in S'(\mathbb{R}^2) : \int_{\mathbb{R}^2} (1 + |\xi|^2)^s |\widehat{u}(\xi)|^2 d\xi < \infty \right\}, \;\; s \in \mathbb{R}.$$

Here, $\widehat{u}(\xi)$ is the Fourier image of $u(x)$ defined by

$$\widehat{u}(\xi) = \frac{1}{2\pi} \int_{\mathbb{R}^2} e^{-ix\xi} u(x) dx.$$

Let

$$\begin{aligned} H^s(S_n) &:= \left\{ v = \tilde{v}|_{S_n} \;:\; \tilde{v} \in H^s(\mathbb{R}^2) \right\}, \\ \|v\|_{H^s(S_n)} &:= \inf_{\substack{\tilde{v} \in H^s(\mathbb{R}^2) \\ \tilde{v}|_{S_n} = v}} \|\tilde{v}\|_{H^s(\mathbb{R}^2)}\,. \end{aligned}$$

Now, let $X_0 = L^2(S_n), X_1 = H^s(S_n)$ for $\frac{1}{2} < s < 1$ and $X_2 = W_2^1(S_n)$ in Lemma 4.4. That $X_2 \hookrightarrow X_1$ is compact follows from the Sobolev compact embedding theorem (see, e.g., [1, Ch. VII] or [20, § 1.4.6]). Thus we have the following lemma:

Lemma 4.5. *Let $\varepsilon > 0$ be given. Then there exists a constant $C_1 > 0$ such that*

$$\int_n^{n+1} |u(x_1,a)|^2\,dx_1 + \int_n^{n+1} |u(x_1,0)|^2\,dx_1 \le C_1\left(\mathcal{E}_{S_n}[u] - \lambda_1\int_{S_n} |u(x)|^2\,dx\right),$$
$$\forall u \in W_2^1(S_n),\ u \perp u_1. \tag{24}$$

Proof. In this proof, we make use of (22) and Lemma 4.4. For $s > \frac{1}{2}$, the trace theorem and Lemma 4.4 imply

$$\begin{aligned}
&\int_n^{n+1} |u(x_1,a)|^2 dx_1 + \int_n^{n+1} |u(x_1,0)|^2 dx_1 \le C_s\|u\|_{X_1}\\
&\le C_s\left(\varepsilon\left(\int_{S_n} |\nabla u(x)|^2 dx + \int_{S_n} |u(x)|^2 dx\right) + C(\varepsilon)\int_{S_n} |u(x)|^2 dx\right)\\
&= C_s\varepsilon\int_{S_n} |\nabla u(x)|^2 dx + C_s(\varepsilon + C(\varepsilon))\int_{S_n} |u(x)|^2 dx\\
&= C_s\varepsilon\Big(\mathcal{E}_n[u] - \lambda_1\int_{S_n} |u(x)|^2 dx - \beta\int_n^{n+1} |u(x_1,a)|^2\,dx_1\\
&+\alpha\int_n^{n+1} |u(x_1,0)|^2\,dx_1 + \lambda_1\int_{S_n} |u(x)|^2\,dx\Big) + C_s(\varepsilon + C(\varepsilon))\int_{S_n} |u(x)|^2\,dx\\
&\le C_s\varepsilon\left(\mathcal{E}_{S_n}[u] - \lambda_1\int_{S_n} |u(x)|^2\,dx\right)\\
&+C_s\varepsilon\max\{|\beta|,|\alpha|\}\left(\int_n^{n+1} |u(x_1,a)|^2\,dx_1 + \int_n^{n+1} |u(x_1,0)|^2\,dx_1\right)\\
&+ C_s\left(\varepsilon(\lambda_1+1) + C(\varepsilon)\right)\int_{S_n} |u(x)|^2\,dx\,.
\end{aligned}$$

Take $\varepsilon \le \frac{1}{2C_s\max\{|\beta|,|\alpha|\}}$. Then

$$\begin{aligned}
&\int_n^{n+1} |u(x_1,a)|^2\,dx_1 + \int_n^{n+1} |u(x_1,0)|^2\,dx_1\\
&\le C_s\varepsilon\left(\mathcal{E}_{S_n}[u] - \lambda_1\int_{S_n} |u(x)|^2\,dx\right) + \frac{1}{2}\int_n^{n+1} |u(x_1,a)|^2\,dx_1\\
&+\frac{1}{2}\int_n^{n+1} |u(x_1,0)|^2\,dx_1 + C_s\left(\varepsilon(\lambda_1+1) + C(\varepsilon)\right)\int_{S_n} |u(x)|^2\,dx.
\end{aligned}$$

Hence (22) yields

$$
\begin{aligned}
&\int_n^{n+1} |u(x_1,a)|^2\,dx_1 + \int_n^{n+1} |u(x_1,0)|^2\,dx_1 \\
&\le 2C_s\varepsilon\left(\mathcal{E}_{S_n}[u] - \lambda_1 \int_{S_n} |u(x)|^2\,dx\right) + 2C_s\left(\varepsilon(\lambda_1+1) + C(\varepsilon)\right)\int_{S_n} |u(x)|^2\,dx \\
&\le C_s\left(2\varepsilon + \frac{2}{\lambda_2-\lambda_1}(\varepsilon(\lambda_1+1)+C(\varepsilon))\right)\left(\mathcal{E}_{S_n}[u] - \lambda_1\int_{S_n}|u(x)|^2\,dx\right) \\
&= C_1\left(\mathcal{E}_{S_n}[u] - \lambda_1\int_{S_n}|u(x)|^2\,dx\right),
\end{aligned}
$$

where

$$
C_1 := C_s\left(2\varepsilon + \frac{2}{\lambda_2-\lambda_1}(\varepsilon(\lambda_1+1)+C(\varepsilon))\right).
$$

□

As a consequence of Lemma 4.5 and (22) we have the following Lemma

Lemma 4.6. *There exists a constant $C_2 > 0$ such that*

$$
\int_{S_n} |\nabla u(x)|^2\,dx \le C_2\left(\mathcal{E}_{S_n}[u] - \lambda_1\int_{S_n}|u(x)|^2\,dx\right), \quad \forall u \in W_2^1(S_n),\ u \perp u_1. \tag{25}
$$

Proof.

$$
\begin{aligned}
\int_{S_n} |\nabla u(x)|^2\,dx &= \mathcal{E}_{S_n}[u] - \lambda_1\int_{S_n}|u(x)|^2\,dx + \lambda_1\int_{S_n}|u(x)|^2\,dx \\
&- \beta\int_n^{n+1}|u(x_1,a)|^2\,dx_1 + \alpha\int_n^{n+1}|u(x_1,0)|^2\,dx_1 \\
&\le (1 + C_1\max\{|\alpha|,|\beta|\})\left(\mathcal{E}_{S_n}[u] - \lambda_1\int_{S_n}|u(x)|^2\,dx\right) \\
&+ \lambda_1\int_{S_n}|u(x)|^2\,dx \\
&\le (1 + C_1\max\{|\alpha|,|\beta|\})\left(\mathcal{E}_{S_n}[u] - \lambda_1\int_{S_n}|u(x)|^2\,dx\right) \\
&+ \frac{\max\{0,\lambda_1\}}{\lambda_2-\lambda_1}\left(\mathcal{E}_{S_n}[u] - \lambda_1\int_{S_n}|u(x)|^2\,dx\right) \\
&= \left(1 + C_1\max\{|\alpha|,|\beta|\} + \frac{\max\{0,\lambda_1\}}{\lambda_2-\lambda_1}\right)\left(\mathcal{E}_{S_n}[u] - \lambda_1\int_{S_n}|u(x)|^2\,dx\right) \\
&= C_2\left(\mathcal{E}_{S_n}[u] - \lambda_1\int_{S_n}|u(x)|^2\,dx\right),
\end{aligned}
$$

where

$$C_2 := 1 + C_1 \max\{|\alpha|, |\beta|\} + \frac{\max\{0, \lambda_1\}}{\lambda_2 - \lambda_1}\,.$$

□

Let $\mathcal{H}_1 := PW_2^1(S)$ and $\mathcal{H}_2 := (I-P)W_2^1(S)$, where

$$Pu(x) := \left(\int_0^a u(x)\overline{u_1}(x_2)\,dx_2\right) u_1(x_2) = w(x_1)u_1(x_2)\,, \qquad \forall u \in W_2^1(S) \tag{26}$$

and

$$w(x_1) := \int_0^a u(x)\overline{u_1}(x_2)\,dx_2\,.$$

Then P is a projection since $P^2 = P$.

Lemma 4.7. *For all* $u \in W_2^1(S)$, $\langle (I-P)u(x_1,.), u_1\rangle_{L^2(0,a)} = 0$ *for almost all* $x_1 \in \mathbb{R}$.

Proof. Since $Pu = \langle u(x_1,.), u_1\rangle_{L^2(0,a)} u_1$, then

$$\begin{aligned}
\langle (I-P)u(x_1,.), u_1\rangle_{L^2(0,a)} &= \langle u(x_1,.) - \langle u(x_1,.), u_1\rangle_{L^2(0,a)} u_1, u_1\rangle_{L^2(0,a)} \\
&= \langle u(x_1,.), u_1\rangle_{L^2(0,a)} - \langle u(x_1,.), u_1\rangle_{L^2(0,a)} \langle u_1, u_1\rangle_{L^2(0,a)} \\
&= \langle u(x_1,.), u_1\rangle_{L^2(0,a)} - \langle u(x_1,.), u_1\rangle_{L^2(0,a)} = 0.
\end{aligned}$$

□

Lemma 4.8. *For all* $v \in \mathcal{H}_1,\ \tilde{v} \in \mathcal{H}_2$, $\langle v, \tilde{v}\rangle_{L^2(S)} = 0$ *and* $\langle v_{x_1}, \tilde{v}_{x_1}\rangle_{L^2(S)} = 0$.

Proof.

$$\begin{aligned}
\langle \tilde{v}, v\rangle_{L^2(S)} &= \int_S (I-P)u(x).\overline{w(x_1)u_1(x_2)}\,dx \\
&= \int_{\mathbb{R}} \overline{w(x_1)} \left(\int_0^a u(x)\overline{u_1(x_2)}\,dx_2\right) dx_1 \\
&- \int_{\mathbb{R}} \overline{w(x_1)} \left[\left(\int_0^a u\overline{u_1(x_2)}\,dx_2\right)\left(\int_0^a u_1(x_2)\overline{u_1(x_2)}\,dx_2\right)\right] dx_1 \\
&= \int_{\mathbb{R}} \overline{w(x_1)} \left(\int_0^a u(x)\overline{u_1(x_2)}\,dx_2\right) dx_1 \\
&- \int_{\mathbb{R}} \overline{w(x_1)} \left[\left(\int_0^a u(x)\overline{u_1(x_2)}\,dx_2\right) \|u_1\|^2\right] dx_1 \\
&= \int_{\mathbb{R}} \overline{w(x_1)} \left(\int_0^a u(x)\overline{u_1(x_2)}\,dx_2\right) dx_1 \\
&- \int_{\mathbb{R}} \overline{w(x_1)} \left(\int_0^a u(x)\overline{u_1(x_2)}\,dx_2\right) dx_1 = 0.
\end{aligned}$$

Since $(Pw)_{x_1} = Pw_{x_1}$ for every $w \in W_2^1(S)$, one has for all $v \in \mathcal{H}_1$ and $\tilde{v} \in \mathcal{H}_2$,

$$v_{x_1} \in PL^2(S),\ \tilde{v}_{x_1} \in (I-P)L^2(S),$$

and hence it follows from the above that

$$\langle v_{x_1}, \tilde{v}_{x_1} \rangle_{L^2(S)} = 0.$$

□

Lemma 4.9. *Let*

$$\mathcal{E}_S[u] := \int_S |\nabla u(x)|^2\, dx + \beta \int_{\mathbb{R}} |u(x_1,a)|^2\, dx_1 - \alpha \int_{\mathbb{R}} |u(x_1,0)|^2\, dx_1, \quad \forall u \in W_2^1(S).$$

Then

$$\mathcal{E}_S[u] = \mathcal{E}_S[v] + \mathcal{E}_S[\tilde{v}], \quad \forall u = v + \tilde{v},\ v \in \mathcal{H}_1\,, \quad \tilde{v} \in \mathcal{H}_2\,.$$

Proof.

$$\begin{aligned}
\langle \tilde{v}_{x_2}, v_{x_2} \rangle_{L^2(S)} &= \int_S \frac{\partial}{\partial x_2}(I-P)u(x) \frac{\partial}{\partial x_2}(\overline{w(x_1)u_1(x_2)})dx \\
&= \int_{\mathbb{R}} \overline{w(x_1)} \left[\int_0^a \frac{\partial}{\partial x_2}(I-P)u(x) \frac{\partial}{\partial x_2}(\overline{u_1(x_2)})\, dx_2 \right] dx_1.
\end{aligned}$$

Integration by parts and Lemma 4.7 give

$$\begin{aligned}
\langle \tilde{v}_{x_2}, v_{x_2} \rangle_{L^2(S)} &= \int_{\mathbb{R}} \overline{w(x_1)}(I-P)u(x_1,a) \frac{\partial}{\partial x_2}\overline{u_1(a)}\, dx_1 \\
&- \int_{\mathbb{R}} \overline{w(x_1)}(I-P)u(x_1,0) \frac{\partial}{\partial x_2}\overline{u_1(0)}\, dx_1 \\
&+ \int_{\mathbb{R}} \overline{w(x_1)} \left(\underbrace{\lambda_1 \int_0^a (I-P)u(x)\overline{u_1(x_2)}\, dx_2}_{=0} \right) dx_1 \\
&= -\beta \int_{\mathbb{R}} \overline{w(x_1)}(I-P)u(x_1,a)\overline{u_1(a)}dx_1 \\
&+ \alpha \int_{\mathbb{R}} \overline{w(x_1)}(I-P)u(x_1,0)\overline{u_1(0)}dx_1.
\end{aligned}$$

Thus, this together with Lemma 4.8 yield

$$\begin{aligned}
\mathcal{E}_S(\tilde{v}, v) &= \int_S \nabla\tilde{v}\overline{\nabla v}\, dx + \beta \int_{\mathbb{R}} \overline{w(x_1)}(I-P)u(x_1,a)\overline{u_1(a)}dx_1 \\
&- \alpha \int_{\mathbb{R}} \overline{w(x_1)}(I-P)u(x_1,0)\overline{u_1(0)}dx_1 \\
&= -\beta \int_{\mathbb{R}} \overline{w(x_1)}(I-P)u(x_1,a)\overline{u_1(a)}dx_1 \\
&+ \alpha \int_{\mathbb{R}} \overline{w(x_1)}(I-P)u(x_1,0)\overline{u_1(0)}dx_1 \\
&+ \beta \int_{\mathbb{R}} \overline{w(x_1)}(I-P)u(x_1,a)\overline{u_1(a)}dx_1 \\
&- \alpha \int_{\mathbb{R}} \overline{w(x_1)}(I-P)u(x_1,0)\overline{u_1(0)}dx_1 = 0.
\end{aligned}$$

This means that for all $u \in W_2^1(S)$

$$\mathcal{E}_S[u] = \mathcal{E}_S[v] + \mathcal{E}_S[\tilde{v}], \quad \forall u = v + \tilde{v},\ v \in \mathcal{H}_1,\ \ \tilde{v} \in \mathcal{H}_2.$$

□

5 Proof of Threorem 3.1

Let

$$\begin{aligned}
\mathcal{E}_S[u] &:= \int_S |\nabla u(x)|^2\, dx - \alpha \int_{\mathbb{R}} |u(x_1,0)|^2\, dx_1 + \beta \int_{\mathbb{R}} |u(x_1,a)|^2\, dx_1, \\
\mathrm{Dom}(\mathcal{E}_S) &= W_2^1(S).
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{E}_{\lambda_1,\mu,S}[u] &:= \mathcal{E}_S[u] - \lambda_1 \int_S |u(x)|^2\, dx - \int_{\overline{S}} V(x)|u(x)|^2\, d\mu(x), \\
\mathrm{Dom}(\mathcal{E}_{\lambda_1,\mu,S}) &= W_2^1(S) \cap L^2\left(\overline{S}, V d\mu\right).
\end{aligned}$$

Then one has

$$N_-\left(\mathcal{E}_{\lambda_1,\mu,S}\right) \le N_-(\mathcal{E}_{1,2\mu}) + N_-(\mathcal{E}_{2,2\mu}) \tag{27}$$

where $\mathcal{E}_{1,2\mu}$ and $\mathcal{E}_{2,2\mu}$ are the restrictions of the form $\mathcal{E}_{\lambda_1,2\mu,S}$ to the spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ respectively. We start by estimating the first term in the right-hand side of (27).

Recall that for all $u \in \mathcal{H}_1\,,\ u(x) = w(x_1)u_1(x_2)$ (see (26)). Let I be an arbitrary interval in $\mathbb{R}$ and let

$$\nu(I) := \int_I \int_0^a V(x)|u_1(x_2)|^2\, d\mu(x).$$

Then

$$\begin{aligned}\int_{\overline{S}} V(x)|u(x)|^2\, d\mu(x) &= \int_{\mathbb{R}} \int_0^a V(x)|w(x_1)u_1(x_2)|^2\, d\mu(x) \\ &= \int_{\mathbb{R}} |w(x_1)|^2\, d\nu(x_1) = \int_{\mathbb{R}} |w(x_1)|^2\, d\nu(x_1).\end{aligned}$$

On the subspace $\mathcal{H}_1$, one has

$$\begin{aligned}&\int_S \big(\mid \nabla u(x) \mid^2 - \lambda_1 \mid u(x) \mid^2\big)\, dx + \beta \int_{\mathbb{R}} \mid u(x_1, a) \mid^2\, dx_1 \\ &-\alpha \int_{\mathbb{R}} \mid u(x_1, 0) \mid^2\, dx_1 - 2 \int_{\overline{S}} V(x) \mid u(x) \mid^2\, d\mu(x) \\ &= \int_{\mathbb{R}} \mid w'(x_1) \mid^2 \left(\int_0^a \mid u_1(x_2) \mid^2\, dx_2\right) dx_1 \\ &+ \int_{\mathbb{R}} \mid w(x_1) \mid^2 \left(\int_0^a \mid u_1'(x_2) \mid^2\, dx_2\right) dx_1 \\ &-\lambda_1 \int_{\mathbb{R}} \mid w(x_1) \mid^2 \left(\int_0^a \mid u_1(x_2) \mid^2\, dx_2\right) dx_1 \\ &+\beta \int_{\mathbb{R}} \mid w(x_1)u_1(a) \mid^2\, dx_1 - \alpha \int_{\mathbb{R}} \mid w(x_1)u_1(0) \mid^2\, dx_1 \\ &-2 \int_{\mathbb{R}} |w(x_1)|^2\, d\nu(x_1).\end{aligned}$$

But

$$\begin{aligned}&\int_{\mathbb{R}} \mid w(x_1) \mid^2 \left(\int_0^a \mid u_1'(x_2) \mid^2\, dx_2\right) dx_1 \\ &= \lambda_1 \int_{\mathbb{R}} \mid w(x_1) \mid^2 \left(\int_0^a \mid u_1(x_2) \mid^2\, dx_2\right) dx_1 \\ &-\beta \int_{\mathbb{R}} \mid w(x_1)u_1(a) \mid^2\, dx_1 + \alpha \int_{\mathbb{R}} \mid w(x_1)u_1(0) \mid^2\, dx_1,\end{aligned}$$

which implies

$$\int_S \left(\mid \nabla u(x) \mid^2 - \lambda_1 \mid u(x) \mid^2\right) dx + \beta \int_{\mathbb{R}} \mid u(x_1, a) \mid^2 dx_1$$
$$-\alpha \int_{\mathbb{R}} \mid u(x_1, 0) \mid^2 dx_1 - 2 \int_{\overline{S}} V(x) \mid u(x) \mid^2 d\mu(x)$$
$$= \|u_1\|^2 \int_{\mathbb{R}} \mid w'(x_1) \mid^2 dx_1 - 2 \int_{\mathbb{R}} |w(x_1)|^2 d\nu(x_1)$$
$$= \int_{\mathbb{R}} \mid w'(x_1) \mid^2 dx_1 - 2 \int_{\mathbb{R}} |w(x_1)|^2 d\nu(x_1). \tag{28}$$

Hence, we have the following one-dimensional Schrödinger operator

$$-\frac{d^2}{dx_1^2} - 2\nu \quad \text{on } L^2(\mathbb{R})\,.$$

Let

$$\begin{aligned} \mathcal{E}_{1,2\nu}[w] &:= \int_{\mathbb{R}} |w'(x_1)|^2 dx_1 - 2 \int_{\mathbb{R}} |w(x_1)|^2 d\nu(x_1), \\ \mathrm{Dom}(\mathcal{E}_{1,2\nu}) &= W_2^1(\mathbb{R}) \cap L^2(\mathbb{R}, d\nu)\,, \end{aligned}$$

$$\begin{aligned} F_n &:= \int_{I_n} |x_1|\, d\nu(x_1), \quad n \neq 0, \\ F_0 &:= \int_{I_0} d\nu(x_1). \end{aligned}$$

Then

$$N_-\left(\mathcal{E}_{1,2\nu}\right) \le 1 + 7.61 \sum_{\{F_n > 0.046,\ n \in \mathbb{Z}\}} \sqrt{F_n} \tag{29}$$

(see [13, (2.42)], see also the estimate before (39) in [26]). To write the above estimate in terms of the original measure, let

$$\begin{aligned} \mathcal{F}_n &:= \int_{I_n} \int_0^a |x_1| V(x) |u_1(x_2)|^2 d\mu(x), \quad n \neq 0, \\ \mathcal{F}_0 &:= \int_{I_0} \int_0^a V(x) |u_1(x_2)|^2 d\mu(x). \end{aligned}$$

Then $F_n = \mathcal{F}_n$. Hence

$$N_-\left(\mathcal{E}_{1,2\mu}\right) \le 1 + 7.16 \sum_{\{\mathcal{F}_n > 0.046,\ n \in \mathbb{Z}\}} \sqrt{\mathcal{F}_n}\,. \tag{30}$$

Next, we consider the subspace $\mathcal{H}_2 \subset W_2^1(S)$. By (22) and (25), one has

$$\|u\|^2_{W_2^1(S_n)} \le \left(\frac{1}{\lambda_2 - \lambda_1} + C_2\right)\left(\mathcal{E}_{S_n}[u] - \lambda_1 \int_{S_n} |u(x)|^2 dx\right) \tag{31}$$

for all $u \in W_2^1(S_n),\ u \perp u_1$.
Let $S_n := (n, n+1) \times (0, a)$, $n \in \mathbb{Z}$ with $\mu(\overline{S_n}) > 0$ be the set G in Lemma 4.3 and S_n^* be defined as above (see the paragraph after Corollary 4.2). For each n, S_n^* intersects not more than N_0 rectangles to the left of S_n and N_0 rectangles to right of S_n, where $N_0 \in \mathbb{N}$ depends only on a and θ_0 in Corollary 4.2. (It is not difficult to see that the side length of S_n^* is less than or equal to $3\sqrt{a^2+1}$ and hence $\left[3\sqrt{2}\sqrt{a^2+1}\right] + 1$ provides an upper estimate for N_0.) Then (17) implies

$$\begin{aligned}
\mu(S_n^*) &\le \sum_{j=n-N_0}^{n+N_0} \mu\left(\overline{S_j}\right) \\
&= \mu\left(\overline{S_{n-N_0}}\right) + ... + \mu\left(\overline{S_{n-1}}\right) + \mu\left(\overline{S_n}\right) + \mu\left(\overline{S_{n+1}}\right) + ... + \mu\left(\overline{S_{n+N_0}}\right) \\
&\le \left(\frac{1}{c_2^{N_0}} + ... + \frac{1}{c_2}\right)\mu\left(\overline{S_n}\right) + \mu\left(\overline{S_n}\right) + \left(\frac{1}{c_2} + ... + \frac{1}{c_2^{N_0}}\right)\mu\left(\overline{S_n}\right) \\
&= \left(2\left(\frac{1}{c_2} + ... + \frac{1}{c_2^{N_0}}\right) + 1\right)\mu\left(\overline{S_n}\right) \\
&= \kappa_0 \mu\left(\overline{S_n}\right),
\end{aligned}$$

where

$$\kappa_0 := 2\left(\frac{1}{c_2} + ... + \frac{1}{c_2^{N_0}}\right) + 1\,.$$

Now it follows from Lemma 4.3 that for any $V \in L_{\mathcal{B}}(\overline{S_n}, \mu)$, $V \ge 0$

$$\int_{\overline{S_n}} V(x)|u(x)|^2 d\mu(x) \le C_0 m^{-1} \|V\|_{\mathcal{B}, \overline{S_n}, \mu} \|u\|^2_{W_2^1(S_n)}$$

for all $u \in W_2^1(S_n) \cap C(\overline{S_n})$ satisfying the m_0 orthogonality conditions in Lemma 4.3, where the constant C_0 is independent of V, m, and n. Hence (31) implies

$$\int_{\overline{S_n}} V(x)|u(x)|^2 d\mu(x) \le C_3 m^{-1} \|V\|_{\mathcal{B}, \overline{S_n}, \mu} \left(\mathcal{E}_{S_n}[u] - \lambda_1 \int_{S_n} |u(x)|^2 dx\right) \tag{32}$$

for all $u \in W_2^1(S_n) \cap C(\overline{S_n})$, $u \perp u_1$ satisfying the m_0 orthogonality conditions, where

$$C_3 := C_0\left(\frac{1}{\lambda_2 - \lambda_1} + C_2\right).$$

Let

$$\mathcal{E}_{2,2\mu,S_n}[u] := \mathcal{E}_{S_n}[u] - \lambda_1 \int_{S_n} |u(x)|^2\, dx - 2\int_{\overline{S_n}} V(x)|u(x)|^2\, d\mu(x),$$
$$\mathrm{Dom}(\mathcal{E}_{2,2\mu,S_n}) = (I-P)W_2^1(S_n) \cap L^2\left(\overline{S_n}, V d\mu\right) \tag{33}$$

(see (21)). Taking $m = \left[2C_3\|V\|_{\mathcal{B},\overline{S_n},\mu}\right] + 1$ in (32), one has

$$N_-\left(\mathcal{E}_{2,2\mu,S_n}\right) \le C_4\|V\|_{\mathcal{B},\overline{S_n},\mu} + 2, \qquad \forall V \ge 0 \tag{34}$$

where $C_4 := 2C_3$ (see [13, Lemma 3.2.14]). Again, taking $m = 1$ (and $m_0 = 0$; see Lemma 4.3) in (32), we get

$$2\int_{\overline{S_n}} V(x)|u(x)|^2\, d\mu(x) \le C_4\|V\|_{\mathcal{B},\overline{S_n},\mu}\left(\mathcal{E}_{S_n}[u] - \lambda_1\int_{S_n} |u(x)|^2 dx\right),$$

for all $u \in W_2^1(S_n) \cap C(\overline{S_n})$, $u \perp u_1$. If $\|V\|_{\mathcal{B},\overline{S_n},\mu} \le \frac{1}{C_4}$, then

$$N_-\left(\mathcal{E}_{2,2\mu,S_n}\right) = 0\,.$$

Otherwise, (34) implies

$$N_-\left(\mathcal{E}_{2,2\mu,S_n}\right) \le C_5\|V\|_{\mathcal{B},\overline{S_n},\mu}\,,$$

where $C_5 := 3C_4$.

Let $M_n = \|V\|_{\mathcal{B},\overline{S_n},\mu}$ (see Section 3). Then for any $c \le \frac{1}{C_4}$, the variational principle (see (6)) implies

$$N_-\left(\mathcal{E}_{2,2\mu}\right) \le C_5 \sum_{\{M_n > c,\, n\in\mathbb{Z}\}} M_n, \quad \forall V \ge 0\,. \tag{35}$$

Thus (27), (30) and (35) imply (18).

6 Concluding remarks

Remark 6.1. Recall that a sequence $\{a_n\}$ belongs to the "weak l_1-space" (Lorentz space) $l_{1,w}$ if the following quasinorm

$$\|\{a_n\}\|_{1,w} = \sup_{s>0}\left(s\ \mathrm{card}\{n\ :\ |a_n| > s\}\right) \tag{36}$$

is finite. It is a quasinorm in the sense that it satisfies the weak version of the triangle inequality:

$$\|\{a_n\} + \{b_n\}\|_{1,w} \le 2\left(\|\{a_n\}\|_{1,w} + \|\{b_n\}\|_{1,w}\right)$$

(see, e.g., [10] for more details).

Theorem 6.2. (cf. [26, Theorem 9.2]) *Let $V \geq 0$. If $N_-(\mathcal{E}_{\lambda_1,\gamma\mu,S}) = O(\gamma)$ as $\gamma \longrightarrow +\infty$, then $\|\mathcal{F}_n\|_{1,w} < \infty$.*

Proof. Consider the function

$$w_n(x_1) := \begin{cases} 0, & x_1 \leq 2^{n-2} \text{ or } x_1 \geq 2^{n+1}, \\ 4(x_1 - 2^{n-2}), & 2^{n-2} < x_1 < 2^{n-1}, \\ 2^n, & 2^{n-1} \leq x_1 \leq 2^n, \\ 2^{n+1} - x_1, & 2^n < x_1 < 2^{n+1}, \end{cases}$$

$n > 0$. Let $v_n(x) = w_n(x_1)u_1(x_2)$. Then by a computation similar to the one leading to (28) we get

$$\begin{aligned} \mathcal{E}_S[v_n] &- \lambda_1 \int_S |v_n(x)|^2\,dx = \int_S \left(|\nabla v_n(x)|^2 - \lambda_1 |v_n(x)|^2\right) dx \\ &- \alpha \int_{\mathbb{R}} |v_n(x_1,0)|^2 dx_1 + \beta \int_{\mathbb{R}} |v_n(x_1,a)|^2 dx_1 = \int_{\mathbb{R}} |w_n'(x_1)|^2 dx_1 \\ &= \int_{2^{n-2}}^{2^{n-1}} |w_n'(x_1)|^2 dx_1 + \int_{2^{n-1}}^{2^n} |w_n'(x_1)|^2 dx_1 + \int_{2^n}^{2^{n+1}} |w_n'(x_1)|^2 dx_1 \\ &= 4 \cdot 2^n + 0 + 2^n = 5 \cdot 2^n. \end{aligned}$$

Now

$$\begin{aligned} \int_{\overline{S}} V(x)|v_n(x)|^2\,d\mu(x) &\geq \int_{2^{n-1}}^{2^n} \int_0^a V(x) 2^{2n} |u_1(x_2)|^2\,d\mu(x) \\ &\geq 2^n \int_{2^{n-1}}^{2^n} \int_0^a |x_1| V(x) |u_1(x_2)|^2\,d\mu(x) \\ &= 2^n \mathcal{F}_n\,. \end{aligned}$$

It follows from the above that $\mathcal{E}_{\lambda_1,\mu,S}[v_n] < 0$ if $\mathcal{F}_n > 5$, $n > 0$.
One can define functions v_n for $n \leq 0$ similarly to the above and extend to them the previous estimate. The fact that v_n and v_k have disjoint supports if $|m - k| \geq 3$ implies that

$$N_-(\mathcal{E}_{\lambda_1,V\mu,S}) \geq \frac{1}{3}\text{card}\{n \in \mathbb{Z} \,:\, \mathcal{F}_n > 5\}$$

(see [26, Theorem 9.1]). If $N_-(\mathcal{E}_{\lambda_1,\gamma\mu,S}) \leq C\gamma$, then

$$\frac{1}{3}\text{card}\{n \in \mathbb{Z} \,:\, \gamma\mathcal{F}_n > 5\} \leq C\gamma\,,$$

which implies

$$\operatorname{card}\left\{n \in \mathbb{Z} \,:\, \mathcal{F}_n > \frac{5}{\gamma}\right\} \le 3C\gamma\,.$$

With $s = \frac{5}{\gamma}$, we have

$$\operatorname{card}\{n \in \mathbb{Z} \,:\, \mathcal{F}_n > s\} \le C_7 s^{-1}, \ \ s > 0,$$

where $C_7 := 15\,C$. □

Remark 6.3. Suppose that $\mu = |\cdot|$, the Lebesgue measure. Then

$$\begin{aligned}
\mathcal{F}_n &= \int_{I_n} |x_1| \left(\int_0^a V(x)|u_1(x_2)|^2 dx_2\right) dx_1, \quad n \ne 0,\\
\mathcal{F}_0 &= \int_{I_0} \left(\int_0^a V(x)|u_1(x_2)|^2 dx_2\right) dx_1.
\end{aligned}$$

Let $J_n := (n, n+1)$, $I := (0, a)$ and write $\| V \|_{\mathcal{B},S_n}$ and $N_-\left(\mathcal{E}_{\lambda_1,S}\right)$ instead of $\| V \|_{\mathcal{B},S_n,|\cdot|}$ and $N_-\left(\mathcal{E}_{\lambda_1,|\cdot|,S}\right)$ respectively. Further, let

$$\mathcal{D}_n := \int_{J_n} \| V \|_{\mathcal{B},I} \ dx_1\,.$$

Then, using [26, Lemma 7.6] in place of our Lemma 4.3, one gets

$$N_-\left(\mathcal{E}_{\lambda_1,S}\right) \le 1 + 7.61 \sum_{\{\mathcal{F}_n > \, 0.046,\, n\in\mathbb{Z}\}} \sqrt{\mathcal{F}_n} + C_8 \sum_{\{\mathcal{D}_n > \, c,\, n\in\mathbb{Z}\}} \mathcal{D}_n, \quad \forall V \ge 0\,. \tag{37}$$

This estimate is stronger than (18). Indeed, suppose that $\|V\|_{(\mathcal{B},S_n)} = 1$. Since $\mathcal{B}(V)$ satisfies the Δ_2-condition, then $\int_{S_n} \mathcal{B}(V(x))\, dx = 1$ (see (9.21) in [17]). Using (9) and (12), we have

$$\begin{aligned}
\mathcal{D}_n = \int_{J_n} \|V\|_{\mathcal{B},I}\, dx_1 &\le 2\int_{J_n} \|V\|_{(\mathcal{B},I)}\, dx_1\\
&\le 2\int_n^{n+1} \left(1 + \int_0^a \mathcal{B}\left(V(x)\right) dx_2\right) dx_1\\
&= 2 + 2\int_{J_n}\int_0^a \mathcal{B}\left(V(x)\right) dx = 4\\
&= 4\|V\|_{(\mathcal{B},S_n)} \le 4\|V\|_{\mathcal{B},S_n}\\
&= 4M_n.
\end{aligned} \tag{38}$$

The scaling $V \longmapsto tV$, $t > 0$, allows one to extend the above inequality to an arbitrary $V \ge 0$.

By the same procedure as the one leading to (56) in [15], one has the following estimate

$$N_-(\mathcal{E}_{\lambda_1,S}) \le 1 + C_9 \left(\| \, (\mathcal{F}_n)_{n\in\mathbb{Z}} \, \|_{1,w} + \|V_*\|_{L_1(\mathbb{R},L_{\mathcal{B}}(I))}\right), \quad \forall V \ge 0, \tag{39}$$

where

$$\begin{aligned} V_* &:= V(x) - G(x_1), \\ G(x_1) &:= \int_0^a V(x)|u_1(x_2)|^2 \, dx_2 \,, \\ \|V_*\|_{L_1(\mathbb{R},L_{\mathcal{B}}(I))} &:= \int_{\mathbb{R}} \|V_*\|_{\mathcal{B},I} \, dx_1 \,. \end{aligned}$$

Estimates (37) and (39) are equivalent to each other but the advantage of the latter is that it separates the contribution to the eigenvalues of $V(x) = V(x_1)$ from that of $V(x) = V(x_2)$. The condition $\|(\mathcal{F}_n)\|_{1,w} < \infty$ is necessary and sufficient for the semi-classical behaviour of the estimate coming from the subspace $\mathcal{H}_1$ (see Theorem 6.2 above). In addition, if $V_* \in L_1(\mathbb{R}, L_{\mathcal{B}}(I))$, then one gets an analogue of [19, Theorem 1.1], i.e.,

$$N_-(\mathcal{E}_{\lambda_1,\gamma\mu,S}) = O(\gamma) \ \text{ as } \ \gamma \longrightarrow \ +\infty$$

if and only if $\mathcal{F}_n \in l_{1,w}$.

Remark 6.4. One can think of the Dirichlet boundary conditions as the limit of the boundary conditions in (2) as α and β tend to infinity. In this case,

$$\lambda_1 = \frac{\pi^2}{a^2}, \ \ u_1(x_2) = \sin\frac{\pi}{a}x_2, \ \text{ and } \lambda_2 = \min\left\{4\frac{\pi^2}{a^2}, \frac{\pi^2}{a^2} + \pi^2\right\} > \lambda_1.$$

Let $X_n := \{u \in W_2^1(S_n) \, : \, u(x_1, 0) = u(x_1, a) = 0\}$. Then for all $u \in X_n, u \perp \sin\frac{\pi}{a}x_2$, one has an analogue of (22)

$$\int_{S_n} |u(x)|^2 \, dx \le \frac{1}{\pi^2} \max\left\{\frac{a^2}{3}, 1\right\} \left(\int_{S_n} |\nabla u(x)|^2 \, dx - \frac{\pi^2}{a^2}\int_{S_n} |u(x)|^2 \, dx\right). \tag{40}$$

Also, similarly to Lemma 4.6, there is a constant $C > 0$ such that

$$\int_{S_n} |\nabla u(x)|^2 \, dx \le C\left(\int_{S_n} |\nabla u(x)|^2 dx - \frac{\pi^2}{a^2}\int_{S_n} |u(x)|^2 \, dx\right), \ \forall u \in X_n, \ u \perp \sin\frac{\pi}{a}x_2. \tag{41}$$

Now, for all $u \in X := \{u \in W_2^1(S) \,:\, u(x_1, 0) = u(x_1, a) = 0\}$, let

$$Pu(x_1, t) := \left(\frac{2}{a}\int_0^a u(x_1, t)\sin\frac{\pi}{a}t\,dt\right)\sin\frac{\pi}{a}x_2\,.$$

Then $P : X \longrightarrow X$ an orthogonal projection (cf. Lemma 4.9). Let $X_1 := PX$ and $X_2 := (I - P)X$. Furthermore, let

$$\begin{aligned} q_{\mu,S}[u] &:= \int_S |\nabla u(x)|^2\,dx - \frac{\pi^2}{a^2}\int_S |u(x)|^2 dx - \int_{\overline{S}} V(x)|u(x)|^2\,d\mu(x), \\ \mathrm{Dom}(q_{\mu,S}) &= X \cap L^2\left(\overline{S}, V d\mu\right). \end{aligned}$$

Then similarly to (27), we have

$$N_-\left(q_{\mu,S}\right) \le N_-(q_{1,2\mu}) + N_-(q_{2,2\mu}) \tag{42}$$

where $q_{1,2\mu}$ and $q_{2,2\mu}$ are the restrictions of the form $q_{2\mu,S}$ to the subspaces X_1 and X_2 respectively. For an arbitrary interval I on $\mathbb{R}$, let

$$\nu(I) := \frac{2}{a}\int_I\int_0^a V(x)\sin^2\frac{\pi}{a}x_2\,d\mu(x).$$

Then on the subspace X_1, a procedure similar to the one leading to (30) gives an estimate for the first term in (42), where in this case $\mathcal{F}_n$ is given by

$$\begin{aligned} \mathcal{F}_n &= \frac{2}{a}\int_{I_n}\int_0^a |x_1|V(x)\sin^2\frac{\pi}{a}x_2\,d\mu(x), \quad n \ne 0, \\ \mathcal{F}_0 &= \frac{2}{a}\int_{I_0}\int_0^a |V(x)\sin^2\frac{\pi}{a}x_2\,d\mu(x). \end{aligned}$$

On the subspace X_2, it follows from (40) and (41) that there is a constant $C' > 0$ such that

$$\|u\|^2_{X_n} \le C'\left(\int_{S_n} |\nabla u(x)|^2\,dx - \frac{\pi^2}{a^2}\int_{S_n} |u(x)|^2 dx\right), \quad \forall u \in X_n,\; u \perp \sin\frac{\pi}{a}x_2.$$

Thus, we obtain, similarly to (35), an estimate for the second term in (42).

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