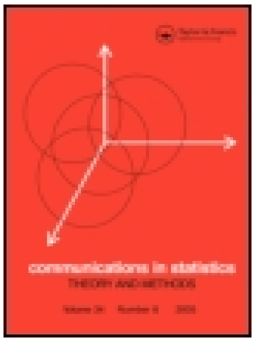

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Group Divisible Designs with Block Size Five from Clatworthy's Table

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Abstract

Clatworthy's table [2] lists thirty seven designs with block size 5 where the number of groups is at the most equal to block size. Mwesigwa, Sarvate and Zhang have generalized one of these designs in a recent paper [16]. In this note we generalize all but one such designs listed in the table. As an aside, we prove that GDD$(n, 4, 5; \frac{9n}{n-1}, 2)$ with intersection pattern $(1, 4)$ does not exist for any n except for $n = 4$.

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1 Introduction

Group Divisible Designs (GDDs) refered here are two associate class Partially Balanced Incomplete Block Designs (PBIBDs) with m groups of size n each, consisting of b binary blocks of size $k \leq mn$. Further we denote the number of treatments (or elements) in the design by $v = mn$. Two treatments from a group are called first associates of each other and occur together λ_1 times and those from differnt groups are called second associates and occur together λ_2 times. A three associate class PBIBD with v treatments consists of three symmetric relations such that any two treatments are either first or second or third associates occuring among the binary blocks accordingly λ_1, λ_2 or λ_3 times respectively. For any undefined terms as well as for the existence results for the families of designs used in the paper, please refer to Stinson [20] and Colbourn and Dinitz [3].

Definition 1 ***Balanced Incomplete Block Design*** $(BIBD)$*: A BIBD is a pair* (V, B) *of a set V with v elements and a set B with b blocks (subsets) of V, each of size* $k < v$*, where each element occurs in exactly r blocks and any two distinct elements of V occur together in exactly* λ *blokcs. A BIBD is represented by its parameters as* $BIBD(v, b, r, k, \lambda)$ *or simply as* $BIBD(v, k, \lambda)$*.*

It is known that the necessary conditions are sufficient for the existence of BIBDs with block size three and four. For block size five with the exception of a BIBD($v = 15, k = 5, \lambda = 2$) the same result hold. See for example, [20].

Definition 2 ***Resolvable Designs****: A block design is* α*-resolvable if the collection of blocks B can be partitioned into, say t, classes such that each element of V is replicated exactly* α *times in each class. Thus for an* α*-resolvable design,* $r = \alpha t$*. For* $\alpha = 1$*, classes are called parallel classes and the design is called a resolvable design.*

1.1 Resolvable BIBD with k=5

We use resolvable balanced incomplete block designs (RBIBD) in some of the constructions. The necessary conditions for the existence of a RBIBD(v, k, λ) are:

- $\lambda(v-1) \equiv 0 \pmod{k-1}$ and

- $v \equiv 0 \pmod{k}$.

For $k = 5$, there are three basic cases, $\lambda = 1, 2$, and 4, since necessary condtions on v depend only on whether $\lambda \equiv 1 \pmod 2$, $\lambda \equiv 2 \pmod 4$ or $\lambda \equiv 0 \pmod 4$.
They are summarized in Abel, Ge, Zhu [1] as follows.

Theorem 1 *Necessary conditions for existence of a RBIBD*$(v, 5, \lambda)$ *are* $\lambda(v-1) \equiv 0 \pmod 4$ *and* $v \equiv 0 \pmod 5$. *For* $\lambda = 1, 2$, *and* 4 *these conditions are sufficient except for* $(v, \lambda) \in \{(10, 4), (15, 2)\}$ *and possibly for the following cases:*

- $\lambda = 1$ *and* $v \in \{45, 225, 345, 465, 645\}$,

- $\lambda = 2$ *and* $v \in \{45, 115, 135, 195, 215, 225, 235, 295, 315, 335,$
  $345, 395\}$ *and*

- $\lambda = 4$ *and* $v \in \{15, 70, 90, 135, 160, 190, 195\}$.

Definition 3 ***Group Divisible Association Scheme*** (*GD*)*: When* v *elements are divided into* m *groups of size* n *each then any two elements from the same group are called first associates and any two elements from the different groups are call second associates. Such an association scheme is called Group Divisible association scheme. Notice that each element has* $n-1$ *first associates and* $n(m-1)$ *second associates.*

Definition 4 ***Group Divisible Design*** (*GDD*)*: A PBIBD with a GD association scheme with* m *groups of size* n *is denoted by GDD*$(n, m,$

$k; \lambda_1, \lambda_2$), if the blocks are of size k, the first associate pairs occur in λ_1 blocks and the second associate pairs occur in λ_2 blocks.

According to their parametric relationships, GDDs are classifed into three classes, (see e.g. [17], p127, Def 8.4.1):

1. Singular Group Divisible Designs (SGD) if $r - \lambda_1 = 0$
2. Semi-regular Group Divisible Designs (SRGD) if $r - \lambda_1 > 0$ and $rk - v\lambda_2 = 0$
3. Regular Group Divisible Designs (RGD) if $r - \lambda_1 > 0$ and $rk - v\lambda_2 > 0$

Remark 1 *For a singular GDD, the necessary condition $r = \lambda_1$ forces that if an element is in the block all first associates must also be in the block and hence the group size divides the block size, hence the block size k must be a composite number. Thus SGDs do not exist for $k = 5$.*

Fu, Rodger and Sarvate [5] and Fu and Rodger [4] completely settled the existence of group divisible designs with $k = 3$, though the necessary and sufficient conditions for GDDs with block size 3 and $\lambda_1 = 0$ were given in Hanani [8]. One can refer to Theorem 4.1 of Ge [6] reworded below.

Theorem 2 *Necessary and sufficient conditions for the existence of a GDD$(n, m, 3; 0, \lambda)$ are*

1. $m \geq 3$
2. $\lambda(m-1)n \equiv 0 \pmod 2$, *and*
3. $\lambda m(m-1)n^2 \equiv 0 \pmod 6$.

To prove necessary conditions are sufficient for the existence of GDDs is especially difficult when the number of groups is less than the block size. For example, the existence of GDDs with $m \leq 4$ is in general a difficult case to solve and even for $k = 4$ not much has been done. Some of

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the work done on $k = 4$ is as follows: Clatworthy [2] has listed eleven GDDs with $k = 4$ and $m = 3$ with replication number atmost 10. Henson and Sarvate [9] generalized two of these designs. Then Rodger and Rogers [18] generalized three more designs from the said list. Subsequently in [19] they gave a generalization of another five from that list. Gao and Ge [7] gave general methods of construction of GDDs with $k = 4$ and also independently generalized all the eleven designs. Hurd and Sarvate [13] constructed GDDs with $k = 4$ using Bhaskar Rao designs and gave necessary and sufficient conditions for the existence when $3 \leq n \leq 8$.

For GDDs with $k = 5$, Hurd, Mishra and Sarvate [11] have given an explicit construction using MOLS of order n, with $m = 2$ or 3 or 6 groups. These designs are not listed in Clatworthy table [2] because their parameter range is beyond that of the table. Hurd, Mishra and Sarvate [12] took in to account the block-group intersection pattern to construct GDDs with $k = 5$ and $m = 2$ groups. Obviously there are only three intersection patterns, viz. $(0, 5), (1, 4)$, and $(2, 3)$ with $m = 2$ and $k = 5$, where a block is said to be of type (a, b) or with intersection pattern (a, b) if there are a treatments from one of the groups and b treatments from another.

For $k = 6$, Keranen and Laffin [14] have constructed GDDs with two groups and block size six. For the block configuration $(s, t) = (3, 3)$, they proved that the necessary conditions are sufficient for the existence of GDD$(n, 2, 6; \lambda_1, \lambda_2)$. Further for GDDs with the configuration $(1, 5)$ they gave examples with minimal or near minimal index for group sizes $n \geq 5$, except for $n = 10, 15, 160$ and 190.

1.2 Block-group intersection

The type of a block is defined by its intersection pattern with groups. The type of a block b of a GDD, is the set $\{|b \bigcap G| : G \text{ is a group of the GDD}\}$. We denote this set as a tuple $(a_1, a_2, \cdots, a_s)$ where $a_1 \leq a_2 \leq \cdots \leq a_s$ and $\Sigma a_i = k$, obviously $s \leq m$. If all blocks of a GDD are of the same type or intersection pattern, we say that the GDD is of that particular intersection pattern. For example,

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if the intersection pattern of a GDD is $(1, 2, 2)$, then all blocks of the GDD are of size 5, and for any given block there are two groups which intersect the block in two elements each and one group intersects the block in one element. In the present paper we focus on GDDs from Clatworthy table [2] with $k = 5$ and $m \leq 5$.

For $k = 5$, there are seven possible intersection patterns: $(1, 1, 1, 1, 1)$, $(1, 1, 1, 2)$, $(1, 1, 3)$, $(1, 2, 2)$, $(1, 4)$, $(2, 3)$ and (5). Not all designs given in Clatworthy's table have all blocks of the same configuration and hence for the purpose of classification we also include a mixed type of GDD. A GDD of type (5) means every block is a subset of one of the groups, hence necessarily $\lambda_2 = 0$. This also mean we have just a collection of BIBDs on individual groups with index λ_1. We will not discuss this type of GDD, besides Clatworthy table [2] does not list any such GDDs. Also, note that if a GDD is of the type $(1, 1, 1, 1, 1)$, then $\lambda_1 = 0$.

There are fortyfive designs in Clatworthy table with $k = 5$. In Section 2 we list these designs according to the intersection patterns of the blocks. In subsequent sections we generalize the constructions of some of these designs with $m \leq 5$. By *generalizing a GDD* we mean to construct a family of designs with certain parameters such that for a specific value we get the design listed in Clatworthy table with the same intersection pattern.

Pattern $(1, 1, 1, 2)$ has been studied by Mwesgwa, Sarvate and Zhang [16], where they have proven that the necessary conditions are sufficient for the existence of such GDDs with four groups except when the group size $n \equiv 0 \pmod 6$.

The following theorem is taken from Ge [6]

Theorem 3 *For $\lambda > 1$, necessary conditions for the existence of a GDD$(n, m, 5; 0, \lambda)$, viz.,*

1. $m \geq 5$,
2. $\lambda(m - 1)n \equiv 0 \pmod 4$ *and*
3. $\lambda m(m - 1)n^2 \equiv 0 \pmod{20}$.

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are also sufficient, except possibly when $\lambda = 2$, $m = 15$ *and either* $n = 9$ *or* $gcd(n, 15) = 1$.

In view of the above result, designs with $\lambda_1 \neq 0$ or $m < 5$ are more interesting candidates to generalize. Hence now we focus more on constructing designs with $\lambda_1 \neq 0$.

2 Seven tables

In this section we list fortyfive designs with $k = 5$ from the Clatworthy's table according to their types. In many instances, Clatworthy lists several solutions for a GDD with the same parameters. For example, R138 has been listed twice in the following tables as it has two solutions where the first solution is of Type $(1, 2, 2)$ and the second solution is of Type $(2, 3)$. Many design parameters are multiples of smaller design parameters given in the table and so are their respective solutions. The comment column gives such information and the section number where the design has been generalized. We have generalized the smallest designs as it automatically implies that it's multiple has been generalized.

Remark 2 *Pattern* $(1, 1, 1, 1, 1)$ *imposes the condition that* λ_1 *must be* 0. *The first thirteen GDDs listed in Table 1 are SR GDDs with* $\lambda_1 = 0$. *For these designs, we have* $r = n\lambda_2$ *and* $b = n^2\lambda_2$ *and parameters are multiples of transversal designs. The last seven designs in Table 1 are regular GDDs with m greater than five. Hence we do not discuss their generalizations in this paper.*

3 Intersection pattern (1,1,1,2)

This pattern imposes a necessay condition on the number of groups m viz., $m > 3$.

3.1 Generalization of R134

Theorem 4 *If a* $GDD(2, m, 3; 0, 1)$ *exists, then a* $GDD(2, m, 5; \lambda_1 = \frac{2(m-1)(m-3)}{3}, \lambda_2 = 3m - 9)$ *exists for* $m \geq 3$.

For $m = 3$, please see Example 1 below. For $m > 3$, we start with a GDD$(2, m, 3; 0, 1)$, which exists for $m \equiv 0, 1 \pmod 3$ (see Theorem 2). It has $b = \frac{2m(m-1)}{3}$ blocks and the replication number r is $(m - 1)$. For any group say $\{x, y\}$, x is in $(m - 1)$ triples and y is in another $m - 1$ triples. As x and y do not appear together in any blocks, the number of blocks containing neither x nor y is $b - 2(m - 1) = \frac{2(m-1)(m-3)}{3}$. We construct blocks of size 5 as follows. Let B be a block of GDD$(2, m, 3; 0, 1)$ which does not contain elements from a group say G then the new block of size 5 is $G \cup B$. Therefore, (x, y) will occur in $\lambda_1 = b - 2(m - 1) = \frac{2(m-1)(m-3)}{3}$ blocks. To enumerate λ_2, we observe the following. Let x and z be second associates and let z be in a group $H = \{z, w\}$. Now except $\{x, z, u\}$, and $\{x, w, v\}$ where u and v are elements from groups other than G and H, remaining $m - 3$ blocks containing x will be used with H to create blocks of size five and hence in these blocks (x, z) will occur $m - 3$ times. Consider z in $\{y, z, s\}$ where s is an elements from the other group than G and H, there are $m - 3$ blocks with z not containing x or y. These blocks will be unioned with G and we will have another $m - 3$ blocks with (x, z). Now the block $\{x, z, u\}$ will be unioned with all groups except G, H and the group containing u, so another $m - 3$ blocks are contributed towards λ_2 and hence $\lambda_2 = 3(m - 3)$. From the construction it is clear that the solution maintains the intersection pattern.

Example 1 *A GDD*$(2, 3, 3; 0, 1)$ *has* $r = 2$ *and* $b = 4$. *For example, let the groups be* $G_1 = \{1, 2\}$, $G_2 = \{3, 4\}$ *and* $G_3 = \{5, 6\}$ *and four blocks of the GDD be,* $\{1, 3, 5\}$,$\{1, 4, 6\}$, $\{2, 3, 6\}$, *and* $\{2, 4, 5\}$. *These parameters and the blocks as per our construction given above in Theorem 4 leads to an empty collections of blocks, which is a GDD*$(2, 3, 5; 0, 0)$.

In theorem 4, plugging $m = 4$ gives R134 as follows:

Example 2 *Consider a GDD*$(2, 4, 3; 0, 1)$ *with* $r = 3$ *and* $b = 8$. *Let the four groups be* $G_1 = \{1, 2\}$, $G_2 = \{3, 4\}$, $G_3 = \{5, 6\}$ *and* $G_4 = \{7, 8\}$ *and the eight blocks be* $B_1 = \{1, 3, 5\}$, $B_2 = \{1, 4, 7\}$, $B_3 = \{1, 6, 8\}$, $B_4 = \{2, 3, 8\}$, $B_5 = \{2, 4, 6\}$, $B_6 = \{2, 5, 7\}$, $B_7 = \{3, 6, 7\}$, $B_8 = \{4, 5, 8\}$. *Note that* $B_1 \cap G_4 = \emptyset, B_2 \cap G_3 = \emptyset, B_3 \cap G_2 = \emptyset, B_4 \cap G_3 = \emptyset, B_5 \cap G_4 = \emptyset, B_6 \cap G_2 = \emptyset, B_7 \cap G_1 = \emptyset, B_8 \cap G_1 = \emptyset$.

Blocks of new GDD$(2,4,5,\lambda_1,\lambda_2)$ *are* $B_1 \cup G_4 = \{1,3,5,7,8\}$, $B_2 \cup G_3 = \{1,4,7,5,6\}$, $B_3 \cup G_2 = \{1,6,8,3,4\}$, $B_4 \cup G_3 = \{2,3,8,5,6\}$, $B_5 \cup G_4 = \{2,4,6,7,8\}$, $B_6 \cup G_2 = \{2,5,7,3,4\}$, $B_7 \cup G_1 = \{3,6,7,1,2\}$, $B_8 \cup G_1 = \{4,5,8,1,2\}$. *We append to each block, the group that is absent. Thus we get the GDD*$(2,4,5;2,3)$ *which is* $R134$*.*

3.2 Generalization of R150

Theorem 5 *A three class partially balanced block design with* $v = 5(2t+1)$*,* $k = 5$*,* $\lambda_1 = 2$*,* $\lambda_2 = 3$*, and* $\lambda_3 = 3t$ *exists for any* $t \geq 1$*.*

Consider a partition of $v = 5(2t+1)$ elements into 5 sets and we call them groups G_1, G_2, G_3, G_4, and G_5 for convenience. We assume that $G_1 = \{a_1, a_2, \cdots, a_{2t+1}\}$, $G_2 = \{b_1, b_2, \cdots, b_{2t+1}\}$, $G_3 = \{c_1, c_2, \cdots, c_{2t+1}\}$, $G_4 = \{d_1, d_2,$
$\cdots, d_{2t+1}\}$, and $G_5 = \{e_1, e_2, \cdots, e_{2t+1}\}$. The blocks are constructed using the following schemes:

$$(G_1,G_1,G_2,G_3,G_5), (G_1,G_1,G_2,G_3,G_4), (G_2,G_2,G_3,G_4,G_1),$$
$$(G_2,G_2,G_3,G_4,G_5), (G_3,G_3,G_4,G_5,G_2), (G_3,G_3,G_4,G_5,G_1),$$
$$(G_4,G_4,G_5,G_1,G_3), (G_4,G_4,G_5,G_1,G_2), (G_5,G_5,G_1,G_2,G_4),$$

and

$$(G_5,G_5,G_1,G_2,G_3).$$

In each of the schemes to construct the blocks we take two elements from G_i and one each from G_j, G_k, G_l as follows:

Recall a Latin square $L = [l_{ij}]$ of order, t, on $\{1,2,\cdots,t\}$ is called an idempotent Latin square if the $l_{ii} = i$ for $i = 1,2,\cdots,t$ and is called a symmetric Latin square if $l_{ij} = l_{ji}$ for all i, and j in $\{1,2,\cdots,t\}$. A Latin square where rows and columns are labeled by $\{1,2,\cdots,t\}$ is refered to as a quasigroup (L,o) where $ioj = l_{ij}$. Let (L,o) be an idempotent symmetric Latin square ($ISLS$), i.e., a quasigroup of order $2t+1$. Idempotent symmetric latin squares of odd orders can be obtained by using the table for addition modulo n and renaming the entries, e.g. see ([15], p6). We use the

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elements of G_1 to denote the rows and columns of the three ISLS with entries from the symbols of G_2, G_3, G_4 respectively. Then a block is given by $\{a_i, a_j, b_{ioj}, c_{ioj}, d_{ioj}\}$ where $i < j$.
It can be seen easily that (a_i, b_i) type pairs occur $3t$ times, (a_i, b_j), for $i \neq j$ occurs three times and (a_i, a_j), for $i \neq j$ occur two times directly from the schemes. For $t = 1$, the PBIBD thus constructed gives the GDD R150. This construction gives the solution that maintains the intersection pattern.

4 Intersection pattern $(1, 1, 3)$

Clatworthy listed only one design in this class, namely $R159$. We give two genralizations of $R159$.

4.1 First generalization of R159

The construction of R159 given in Clatworthy can be better understood when we consider V as $Z_7 \times Z_5$ and observe that the design is obtained by developing two difference sets $\{(1, 0), (2, 0), (4, 0), (0, 1), (0, 4)\}$ and $\{(1, 0), (2, 0), (4, 0), (0, 2), (0, 3)\}$. Hence we have an immediate generalization for $k = 2t + 1$ in the following theorem:

Theorem 6 *If for no nonzero element a, elements a and −a both are in the difference set D for a cyclic BIBD*$(2k + 1, k, \lambda = \frac{k-1}{2})$*, then a GDD*$(2k + 1, 5, k + 2; k − 1, 1)$ *exists with difference family solutions* $(D \times \{0\}) \cup \{(0, 1), (0, 4)\}$ *and* $(D \times \{0\}) \cup \{(0, 2), (0, 3)\}$

The above theorem with $k = 3$ gives $R159$.

Example 3 *It is known that a BIBD*$(11, 5, 2)$ *exists with difference set* $\{1, 3, 4, 5, 9\}$*, hence a GDD*$(11, 5, 7; 4, 1)$ *exists.*

4.2 Second generalization of R159

This generalization of R159 is for any $m = 2s + 1$:

Theorem 7 *A GDD*$(7, 2s+1, 5; 2s, 1)$ *always exists.*

Observe that $\{(1,0),(2,0),(4,0),(0,1),(0,2s)\}$,
$\{(1,0),(2,0),(4,0),(0,2),(0,2s-1)\},\cdots,\{(1,0),(2,0),(4,0),(0,s-1),(0,s+2)\},\{(1,0),(2,0),(4,0),(0,s),(0,s+1)\}$ is the required difference family on groups $\{i\}\times Z_{2s+1}$ for $i=0,1,\cdots,6$. As an example when $s=2$, the difference family produces the two difference sets for R159 as given in the beginning of this subsection.

5 Intersection pattern (1,2,2)

Clatworthy listed four designs in this class.

5.1 Generalization of R139

The GDD$(2,5,5,4,2)$ solution given as R139 can not be generalized to a family of GDD$(n,5,5,4,2)$ with block intersection pattern $(1,2,2)$. This is due to following necessary parametric relationship:

Recall that for $k=5$, a block contributes ten pairs of elements. Say f pairs are first associates and $(10-f)$ are second associates. This will lead to a linear equation

$$\frac{\lambda_1(n-1)}{f}=\frac{\lambda_2 n(m-1)}{(10-f)}, \text{ for a fixed } m, \lambda_1 \text{ and } \lambda_2.$$

This equation will be satisfied by a unique set of parameters. Thus we generalize $R139$ as in the following theorem.

Theorem 8 *There exists a GDD*$(2t, 5, 5; 4t, 2(2t-1))$ *for all positive integers* t.

We create blocks of size 5 using each of the schemes given below:

$$(G_1,G_2,G_4),\ (G_1,G_3,G_5),\ (G_2,G_5,G_3),\ (G_3,G_4,G_2),\ (G_4,G_5,G_1).$$

Using a scheme, say (G_1,G_2,G_4), means the following:

- Let $E_1, E_2, \cdots, E_{2t-1}$ be a 1-factorization of a K_{2t} on G_1.
- Let $F_1, F_2, \cdots, F_{2t-1}$ be a 1-factorization of a K_{2t} on G_2.
- Combine for $a = 1, 2, \cdots, 2t-1$, 1-factors $E_a = \{e_0, e_1, \cdots, e_{t-1}\}$ with F_a=$\{f_0, f_1, \cdots, f_{t-1}\}$ to construct a collection $X_a^j = \{e_i \cup f_{i+j} : i = 0, 1, \cdots, t-1\}$, $j = 0, 1, \cdots, t-1$ of sets of size 4. The subscript $i + j$ is evaluated modulo t.
- Let $G_4 = \{d_0, d_1, \cdots, d_{2t-1}\}$.
- Append elements d_{2j} to each of the 4-set in X_a^j to construct blocks of size 5 of the required GDD for $j = 0, 1, \cdots, t-1$.
- Also append d_{2j+1} to each of the 4-set in X_a^j to construct blocks of size 5 of the required GDD for $j = 0, 1, \cdots, t-1$.

One can check that we have the required GDD after all the schemes are used. It can be done by observing that each group is coming as first or second coordinate of the scheme twice and hence each edge of K_{2t} on a group, meaning each pair of distinct elements of the group occurs $4t$ times. This is because each edge is in one of the 1-factors, for example e_1, is in a set of size 4 in X_a^j for $j = 0, 1, \cdots, t-1$. When appended with d_{2j} and with d_{2j+1} it occures $2t$ times. Simliarly it occurs $2t$ times through the second scheme. That results $\lambda_1 = 4t$.

To observe that λ_2 is $2(2t-1)$, we have to observe that the pairs from distinct groups may arise in two ways. For example, two groups are together once in the first and second coordinates or two groups occur together in a scheme twice, one of the two group in first or second coordinate and the other group in the third coordinate of a scheme. For example, a pair of groups G_1 and G_2 and a pair of groups G_1 and G_4. Similarly one can check that the number of blocks is $10t^2(2t-1)$. The designs thus obtained, maintain the intersection pattern. In the above generalization, $t = 1$ gives R139.

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Example 4 *Theorem 8 for $t = 2$ gives a GDD$(4,5,5;8,6)$. Its blocks can be written according to the group schemes given in the proof of Theorem 8 . Below we are reproducing the first 24 blocks of GDD (4,5,5;8,6) using the scheme (G_1, G_2, G_4). The rest of the blocks can be generated symmetrically according to the schemes given on the groups in the theorem. Note that, unlike Clatworthy's convention, here the groups are $G_{i+1} = \{(4i + j + 1) | i = 0, 1, 2, 3, 4;\ j = 0, 1, 2, 3\}$.*

1	3	1	3	1	3	1	3	1	2	1	2	1	2	1	2
2	4	2	4	2	4	2	4	3	4	3	4	3	4	3	4
5	7	7	5	5	7	7	5	5	6	6	5	5	6	6	5
6	8	8	6	6	8	8	6	7	8	8	7	7	8	8	7
13	13	14	14	15	15	16	16	13	13	14	14	15	15	16	16

1	2	1	2	1	2	1	2
4	3	4	3	4	3	4	3
5	6	6	5	5	6	6	5
8	7	7	8	8	7	7	8
13	13	14	14	15	15	16	16

6 Intersection pattern $(1, 4)$

An obvious necessary condition for a GDD with intersection pattern $(1, 4)$ is that

$$n \geq 4. \tag{6.1}$$

Another condition comes from observing that if a GDD$(n, m, k; \lambda_1, \lambda_2)$ of type $(1, 4)$ exists, then the intersection of blocks containing all first associate pairs of a group G with the group G gives a a BIBD$(n, k-1, \lambda_1)$ on G. In other words:

Theorem 9 *A necessary condition for a GDD$(n, m, k; \lambda_1, \lambda_2)$ to exist with intersection pattern $(1, k-1)$ is the existence of a BIBD$(n, k-1, \lambda_1)$.*

Also as all b blocks are of type $(1, 4)$, each block contains six first associate pairs and four second associate pairs of elements. Thus the total number of the first associate pairs in the whole design is $6b$ and total number of the second associate pairs is $4b$. Hence $6b = m\binom{n}{2}\lambda_1$ and $4b = \binom{m}{2}n^2\lambda_2$ lead to a necessary condition

$$\lambda_1 = \frac{3(m-1)n\lambda_2}{2(n-1)} \tag{6.2}$$

The next necessary condition arises from the replication number, as

$$r = \frac{(n-1)\lambda_1 + (m-1)n\lambda_2}{k-1} = \frac{((n-1)\lambda_1 + (m-1)n\lambda_2)}{4} \tag{6.3}$$

equivalently we have

$$((n-1)\lambda_1 + (m-1)n\lambda_2) \equiv 0 \pmod 4.$$

Clatworthy listed five designs in this class: R133, R135, R143, R146 and R152. Also R135 is a multiple of R133 and R146 is a multiple of R143. Thus we generalize only three designs namely R133, R143 and R152.

6.1 Generalization of R133

Take n copies of an existing BIBD$(n, 4, \lambda)$. For such a design the replication number $r = \frac{\lambda(n-1)}{3}$.To each block of a copy BIBD$(n, 4, \lambda)$ on the elements of G_1, add an element from the second group G_2. Repeat this by switching the roles of G_1 and G_2. These blocks so constructed give a GDD$(n, 2, 5; n\lambda, \frac{2\lambda(n-1)}{3})$.

For $n = 4$, BIBD is a single block with $\lambda = 1$ and we get R133. In general for $n \equiv 1 \pmod 3$, $\frac{n-1}{3}$ must divide λ_2. Taking multiple copies of R133, the necessary conditions are sufficient for the existence of GDD$(4, 2, 5, 4\lambda_2, \lambda_2)$. Further this construction maintains the pattern $(1, 4)$.

6.2 Generalization of R143

As in the case of R133, the necessary conditions are sufficient for the existence of GDD$(4, 3, 5, 4\lambda_2, \lambda_2)$ as well. This family of GDDs can be constructed as follows: Let the groups be G_1, G_2, G_3. Take

$4\lambda_2$ copies of G_i and adjoin each element of the group G_{i+1}, λ_2 times, for $i = 1, 2, 3$ where the subscripts are evaluated using mod 3.

Further consider Equation 6.2 when $\lambda_2 = (n-1)t$ for some positive integer t. In this situation family of GDDs of interest is GDD$(n, 3, 5, \lambda_1 = 3nt, \lambda_2 = (n-1)t)$ as this GDD also generalizes R143 from Clatworthy table, as when $n = 4$, we have $\lambda_2 = 3t$ and $\lambda_1 = 4\lambda_2 = 12t$. . The family can be constructed as follows: It is known that for $n \equiv 0, 1 \pmod 4$, a BIBD$(n, 4, 3)$ exists([3], p127). Take n copies of a BIBD$(n, 4, 3)$ on n elements of group G_i and append an element of group G_{i+1} and as the replication number is $n-1$, we will have $\lambda_2 = n-1$. Taking t copies of such a GDD will give the family GDD$(n, 3, 5; 3nt, (n-1)t)$, for $n \equiv 0, 1 \pmod 4$. Further It is known that BIBD$(n, 4, 6)$ exist for all n, and hence we can use the same method to construct GDD$(n, 3, 5; 6nt', 2(n-1)t')$ for all $t = 2t'$.

6.3 Generalization of R152

The construction of $R152$, i.e., of GDD

$(4, 5, 5; 8, 1)$, can be generalized to GDD$(4, 2s+1, 5; 4s, 1)$ of type $(1, 4)$ for any positive integer s. The resulting GDD is also 5-resolvable as is $R152$, with $b = 4s(2s+1), r = 5s, k = 5, \lambda_1 = 4s, \lambda_2 = 1$. The groups are $G_i = \{i, m+i, 2m+i, 3m+i\}$ for $i = 1, 2, ..., 2s+1$ The blocks are obtained by cyclically developing the 4s initial blocks as in R152: $\{1, m+1, 2m+1, 3m+1, x\}$, where x belongs to $G_2 \cup G_3 \cup \cdots \cup G_{s+1}$. Cycles are of length $(2s+1)$ cycling within the four subsets of elements $\{1, 2, 3, \cdots, 2s+1\}$, $\{m+1, m+2, m+3, \cdots, m+2s+1\}$, $\{2m+1, 2m+2, 2m+3, \cdots, 2m+2s+1\}$, $\{3m+1, 3m+2, 3m+3, \cdots, 3m+2s+1\}$. This construction gives a solution that ratains the intersection type $(1, 4)$. Putting $s = 2$ in this series gives R152. The construction maintains the $(1, 4)$ pattern.

For intersection pattern$(1, 4)$, $m = 4$ is an interesting case. We notice that Clatworthy has not lised any GDDs with intersection pattern of $(1, 4)$ where $m = 4$. We are explore this case as follows:

6.4 The case of m=4

With the number of groups 4 and intersection pattern $(1, 4)$, the necessary condition reduces to

$$\lambda_1 = \frac{9n\lambda_2}{2(n-1)}. \tag{6.4}$$

Hence a GDD with 4 groups must be a GDD$(n, 4, 5; \frac{9n\lambda_2}{2(n-1)}, \lambda_2)$ with $r = \frac{15n\lambda_2}{8}$, if exists. Since $r \geq 15$, Clatworthy may not have listed any design of type $(1, 4)$ with $m = 4$. On the other hand we will show non-existence of a family of GDDs with intersection pattern $(1, 4)$ and 4 groups.

For $\lambda_2 = 2$, a GDD$(4, 4, 5; 12, 2)$ exists on groups G_i, $i = 1, 2, 3, 4$. The set of elements $V = \cup G_i$ with the collection of blocks $G_i \cup \{j\}\ \forall j \in V - G_i$, $i = 1, 2, 3$ and 4.

The above GDD is not a multiple of GDD$(4, 4, 5; 6, 1)$ because GDD$(4, 4, 5; 6, 1)$ does not exist as the necessary condition (6.4) is not satisfied.

Let $\lambda_2 = 2$, then $\lambda_1 = \frac{9n}{(n-1)}$, hence $n - 1$ must divide 9. Therefore possible values of n are $n = 2, 4$ and 10. Now we are interested in the intersection pattern of $(1, 4)$ therefore $n \geq 4$, hence n can not be 2. For $n = 10$, necessary conditions are neither satisfied for GDD$(10, 4, 5; 5, 1)$ and nor for GDD$(10, 4, 5; 10, 2)$. This is in contrast with the case of GDD$(4, 4, 5; 6, 1)$ where the necessary conditions are not satisfied but GDD$(4, 4, 5; 12, 2)$ exists.

Theorem 10 *A GDD*$(n, 4, 5; \frac{9n}{n-1}, 2)$ *with intersection pattern* $(1, 4)$ *does not exists except for* $n = 4$ *when GDD*$(4, 4, 5; 12, 2)$ *exists.*

For $\lambda_2 = 2(n-1)$, a series of GDD$(n, 4, 5; 9n, 2(n-1))$ can be constructed. For $n = 4$, the design is three copies of the GDD$(4, 4, 5; 12, 2)$. Using the necessary condition $(n-1)\lambda_1 + 3n\lambda_2 \equiv 0 \pmod 4$, we have $9n(n-1) + 6n(n-1) \equiv 0 \pmod 4$, i.e., $9n^2 - 9n + 6n^2 - 6n \equiv 0 \pmod 4$, i.e., $n \equiv 0, 1 \pmod 4$. It is known that or $n \equiv 0, 1 \pmod 4$, BIBD$(n, 4, 3)$ exists ([3], p127) with the replication number $n-1$. Use $3n$ copies of BIBD$(n, 4, 3)$ on each group with $3n$ elements of the other three groups to create blocks of size five of the required GDD. Hence we have

Theorem 11 *Necessary conditions are sufficient for the existence of a GDD*$(n, 4, 5; 9tn, 2t(n-1))$ *for all positive integer values of t.*

7 Intersection pattern (2,3)

7.1 Generalization of R137

Theorem 12 *A GDD*$(6t+3, 3, 5; 8t+4, 6t+2)$ *exists for all integers* $t \geq 0$

Start with a RBIBD(6t+3,3,1), say D, which is known to exist for all non negative integers t ([3], p127). D has $3t+1$ parallel classes, say $\pi_1, \pi_2, \cdots, \pi_{3t+1}$. So first consider D on G_1 and a K_{6t+3} on G_2. Recall that K_{6t+3} has $3t+1$ two-factors, say $T_1, T_2, \cdots, T_{3t+1}$. Take union of each triple/block of π_j with each edge of T_j for $j = 1, 2, \cdots 3t+1$ to create blocks of the required GDD of size 5. We repeat the same procedure with G_2 and G_3 and G_3 and G_1. It is is easy to count λ_1 as we are repeating each triple/block of RBIBD$(6t+3, 3, 1)$, $6t+3$ times as $6t+3$ is the number of edges in a two-factor and we are also using a K_{6t+3}: note that each edge of K_{6t+3} will be appended with $2t+1$ triples/blocks of a parallel class. Hence $\lambda_1 = 6t+3+2t+1 = 8t+4$. Counting of λ_2 can be done in a similar fashion. The degree of a vertex in K_{6t+3} is $6t+2$, so an edge containing say an element a of a group G_1, will be appended with the triples of $6t+2$ parallel classes of the BIBD on G_2 and hence $\lambda_2 = 6t+2$. In the above theorem $t = 0$ gives a GDD R137.

7.2 Generalization of R149

Theorem 13 *A family of GDD with parameters* $(6t+3, 5, 5, 8(2t+1), 2(3t+1))$ *exists for positive integers* $t \geq 0$.

Consider $n \equiv 3 \pmod 6$, equivalently let $n = 6t+3$ for some positive integer t. The set of blocks of the required GDD is union of the set of blocks constructed on the following pairs of the five groups: (G_1, G_2), (G_1, G_3), (G_4, G_1), (G_5, G_1), (G_2, G_3), (G_2, G_4), (G_5, G_2), (G_3, G_4), (G_3, G_5), (G_4, G_5) as follows. Given a pair (G_i, G_j) of groups, we construct blocks of size 5, by attaching each pair of G_j with all triples of a parallel class of a RBIBD$(6t + 3, 3, 6t + 3)$ on G_i.

One can check that $\lambda_1 = 8(2t + 1)$ by observing that in the ordered pairs of $G_i's$ above, each group occurs twice as the first entry and twice as the second entry. When a group, say G_i is the first entry, its pairs of distinct elements occur in the blocks of RBIBD$(6t+3, 3, 6t+3)$, hence contribute $2(6t+3)$ towards the count of λ_1. Now each parallel class of the RBIBD$(6t+3, 3, 6t+3)$ has $2t+1$ triples/blocks, therefore any pair of distinct elements of G_i occurs altogether in $2(2t + 1)$ blocks when G_i is the second entry. Therefore $\lambda_1 = 2(6t + 3) + 2(2t + 1) = 8(2t + 1)$.

To count λ_2, observe that any two groups, (G_i, G_j) are together in only one ordered pair. Let x be an element of G_j and let y be an element of G_i. There are $6t + 2$ pairs of elements of G_j containing x. Each pair is attached with all blocks of a parallel class which contain all elements of G_i, in particular y, exactly once and hence the pair (x, y) occurs in the blocks of the required GDD exactly $(6t + 2) = 2(3t + 1) = \lambda_2$ times. This construction maintains the pattern $(2, 3)$. For $t = 0$ we get GDD R149.

8 Intersection pattern Mixed type

Here is a generalization of R141:

8.1 Generalization of R141

There are two non-isomorphic solutions with parameter GDD(5,2,5;5,4) in Clatworthy [2]. We generalize the first solution as follows:

Generalization of the first solution: First if a BIBD$(2v, v, \lambda)$ exists, then a GDD$(v, 2, v; \lambda + t, \lambda)$

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exists. The blocks of the GDD are the blocks of BIBD together with t copies of a partition of 2v elements in two sets of size v: these two sets play the role of the groups. In fact, more generally, if a BIBD(uv, v, λ) exists, then a GDD$(u, v, v; \lambda + t, \lambda)$ exists.

We can generalize R141 in another way where the GDDs produced have block size 5. Suppose a BIBD$(5t, 5, \lambda)$ exists, then a GDD$(5, t, 5, \lambda + s, \lambda)$ exists, by using partition of 5t elements into t groups of size five, the blcoks of the GDD are blocks of the BIBD and s copies of the groups as blocks.

8.2 Generalization of R155 , R156, R157, R158

These designs are from the same family. The construction is as follows: Take the blocks of RBIBD(25,5,1) together with $\lambda_1 - 1$ copies of a parallel class of blocks.

Theorem 14 *If a RBIBD*$(v = mk, k, \lambda)$ *exists, then a GDD*$(k, m, k; \lambda + t, \lambda)$ *exists.*

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Table 1: Intersection pattern $(1, 1, 1, 1, 1)$

Design	v	r	b	m	n	λ_1	λ_2	Comment
SR52	10	4	8	5	2	0	2	Remark 2
SR53	10	6	12	5	2	0	3	Remark 2
SR54	10	8	16	5	2	0	4	2 solutions, duplicate of SR52
SR55	10	10	20	5	2	0	5	Remark 2
SR56	15	6	18	5	3	0	2	Remark 2
SR57	15	9	27	5	3	0	3	Remark 2
SR58	20	4	16	5	4	0	1	Remark 2
SR59	20	8	32	5	4	0	2	2 solutions, duplicate SR58, Remark 2
SR60	25	5	25	5	5	0	1	Remark 2
SR61	25	10	50	5	5	0	2	Duplicate of SR60 Remark 2
SR62	35	7	49	5	7	0	1	Remark 2
SR63	40	8	64	5	8	0	1	Remark 2
SR64	45	9	81	5	9	0	1	Remark 2
R144	12	5	12	6	2	0	2	$m > 5$
R147	12	10	24	6	2	0	4	Duplicate of R144
R153	24	5	24	6	4	0	1	$m > 5$
R154	24	10	48	6	4	0	2	Duplicate of R153
R161	40	9	72	10	4	0	1	$m > 5$
R162	44	10	88	11	4	0	1	$m > 5$
R163	45	10	90	9	5	0	1	$m > 5$

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Table 2: Intersection pattern $(1, 1, 1, 2)$

Design	v	r	b	m	n	λ_1	λ_2	Comment
R134	8	5	8	4	2	2	3	Generalized in Section 3.1
R136	8	10	16	4	2	4	6	Duplicate of R134
R145	12	5	12	4	3	1	2	Generalized in [16]
R148	12	10	24	4	3	2	4	Duplicate of R145
R150	15	10	30	5	3	2	3	Generalized in Section 3.2
R160	39	10	78	13	3	2	1	$m > 5$

Table 3: Intersection pattern $(1, 1, 3)$

Design	v	r	b	m	n	λ_1	λ_2	Comment
R159	35	10	70	5	7	2	1	Generalized in Sections 4.1, 4.2

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Table 4: Intersection pattern $(1, 2, 2)$

Design	v	r	b	m	n	λ_1	λ_2	Comment
R138	9	10	18	3	3	8	4	Duplicate of R137
R139	10	5	10	5	2	4	2	Generalized in Section 5.1
R142	10	10	20	5	2	8	4	Duplicate of R139
R151	18	10	36	9	2	8	2	$m > 5$

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Table 5: Intersection pattern $(1,4)$

Design	v	r	b	m	n	λ_1	λ_2	Comment
R133	8	5	8	2	4	4	2	Generalized in Section 6.1
R135	8	10	16	2	4	8	4	Duplicate of R133
R143	12	5	12	3	4	4	1	Generalized in Section 6.2
R146	12	10	24	3	4	8	2	Duplicate of R143
R152	20	10	40	5	4	8	1	Generalized in Section 6.3

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Table 6: Intersection pattern $(2,3)$

Design	v	r	b	m	n	λ_1	λ_2	Comment
R137	9	5	9	3	3	4	2	Generalized in Section 7.1
R138	9	10	18	3	3	8	4	Duplicate of R137
R149	15	10	30	5	3	8	2	Generalized in Section 7.2

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Table 7: Intersection pattern mixed type

Design	v	r	b	m	n	λ_1	λ_2	Comment
R140	10	7	14	5	2	4	3	Open
R141	10	10	20	2	5	5	4	Generalized in Section 8.1
R155	25	7	35	5	5	2	1	Generalized in Section 8.2
R156	25	8	40	5	5	3	1	Generalized in Section 8.2
R157	25	9	45	5	5	4	1	Generalized in Section 8.2
R158	25	10	50	5	5	5	1	Generalized in Section 8.2

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