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Group Divisible Design $(4, n, n + 1, 4; \lambda_1, \lambda_2)$, for $n \geq 4$

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Abstract: *This work is about Group Divisible Designs (GDDs) of block size four on three groups of different sizes $n_1 = 4$, $n_2 = n$ and $n_3 = n + 1$ where $n \geq 4$. We first establish necessary conditions for the existence of the GDD using relationships between the parameters of the GDD and then prove that these conditions are sufficient for several families of GDDs.*
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I. Introduction

A group divisible design, GDD $(n, m, k; \lambda_1, \lambda_2)$, is a collection of k- element subsets, called blocks, of an nm, V -set where the elements of V are partitioned into m groups of size n each; each point of V appearing in r of the b blocks; each pair of symbols from the same group appearing in exactly λ_1 blocks and each pair of symbols from different groups occurring in λ_2 blocks. Group divisible designs have been studied for their usefulness in statistics, designs of experiments and their important applications in scheduling, group testing and construction of other types of combinatorial designs such as pairwise balanced designs, packings and frames [1]. A GDD is called uniform if all its groups have the same size, otherwise it is called non uniform.

When GDDs are used to construct other combinatorial designs, the groups are preferred to have different sizes, that is, non-uniform GDDs are used to fit in various situations that may arise [1]. Unfortunately, comparing with uniform GDDs, much less is known on the construction of non-uniform ones. One major reason is that no ap- propriate algebraic or geometric structures have been found for the construction. Therefore, the construction of non-uniform GDDs is still a challenging problem.

Non-uniform GDDs of block size three have been studied in [2, 3, 4, 5, 6, 7] while those of block size four have been studied only for $n_1 = 1$ or 2 in [8] and $n_1 = 3$ in [9]. It is therefore still reasonable to focus on group divisible designs of block size four, solving the problem when the design has three groups of different sizes, $n_1 = 4$, $n_2 = n$ and $n_3 = n + 1$ where $n \geq 4$.

Definition 1.1. [10] A balanced incomplete block design, (V, B) is a finite non empty multiset, B of b nonempty subsets (called blocks) of size k of a V -set such that each element (point) in V appears in exactly r of the blocks, every pair of distinct elements of V occurs in λ blocks and $2 \leq k < v$.

Lemma 1.1. [11] In a (v, b, r, k, λ)-BIBD, the parameters satisfy the necessary conditions;

(i) $vr = bk$
(ii) $\lambda (v - 1) = r (k - 1)$

Theorem 1.2. [12] Let v be a positive integer. Then the necessary and sufficient conditions for existence of a $(v, 4, \lambda)$-BIBD are as follows;
(i) If $\lambda \equiv 1, 5 \pmod 6$, then $v = 1, 4 \pmod{12}$
(ii) If $\lambda \equiv 2, 4 \pmod 6$, then $v = 1 \pmod 3$
(iii) If $\lambda \equiv 3 \pmod 6$, then $v = 0, 1 \pmod 4$
(iv) If $\lambda \equiv 0 \pmod 6$, then $v \geq 4$

Definition 1.2. [5] A group divisible design, GDD $(v = n_1 + n_2 + \cdots + n_m, k; \lambda_1, \lambda_2)$ is a collection of k-element subsets (called blocks) of a V -set of symbols where the V -set is partitioned into m groups of sizes $n_1, n_2, ..., n_m$; each pair of symbols from the same group appearing in exactly λ_1 blocks and each pair of symbols from different groups occurring in λ_2 blocks .

Example 1.3. A GDD (3, 3, 4, 4; 2, 2) with groups $G_1 = \{1, 2, 3\}$, $G_2 = \{4, 5, 6\}$ and $G_3 = \{7, 8, 9, 10\}$ has $r_1 = r_2 = r_3 = 6$ and $b = 15$. Its blocks are;

1	1	1	1	1	1	2	2	2	2	3	3	4	4	5
2	2	3	3	4	6	3	3	4	4	5	6	5	5	6
4	7	5	8	7	8	8	5	7	6	8	7	7	9	7
6	9	6	10	9	9	9	9	10	10	10	8	10	10	8

II. GDD $(4, n, n+1, 4; \lambda_1, \lambda_2)$

A group divisible design GDD $(4, n, n+1, 4; \lambda_1, \lambda_2)$ is a GDD with three groups of different size $n_1 = 4$, $n_2 = n$ and $n_3 = n + 1$ where first associate pairs occur in λ_1 blocks, and second associate pairs occur in λ_2 blocks. We establish necessary conditions and prove that they are sufficient for the existence of this GDD.

Difficulties arise in the construction of GDDs when the number of groups is smaller than the block size [13]. A case in point is when the block size is four and the number of groups is three. Earlier results on such constructions include those of Clatworthy [14] who in 1973 listed only 11 GDDs with three groups and block size four up to $r = 10$.

Henson and Sarvate in [15] generalized one of these designs, namely; GDD (8, 3, 4; 2, 1) and proved that the necessary conditions are sufficient for the existence of the GDD. They consequently constructed a new family of group divisible designs $(6s + 2, 3, 4; 2, 1)$ using mutually orthogonal latin squares (MOLS) of order $3s+1$ and $(3s+1,4,2)$-BIBDs for all positive integers, s.

The problem of existence of group divisible designs of block size 4 on three groups can be answered using results in [16] where the authors show that the necessary conditions are sufficient for the existence of odd and even GDD $(n, 3, 4; \lambda_1, \lambda_2)$ for any n, and for all mixed designs except for the minimal index case for group size 5t where $(t \geq 1)$. These GDDs were also studied by Zhu and Ge in [13] who completed the undetermined families of mixed GDDs using two constructions based on idempotent self-orthogonal latin squares and skew room squares.

In all these studies, the authors considered GDDs of block size four on three groups. However, they did not take care of the cases where the three groups are of different sizes.

The existence problem of GDDs when the three groups have different sizes has been studied recently. Hurd and Sarvate in [3] found all ordered pairs (n, λ) of positive integers such that a GDD $(v = 1 + 1 + n, 3; 1, \lambda)$ exists and all ordered triples $(n, \lambda_1, \lambda_2)$ of positive integers with $\lambda_1 < \lambda_2$ such that a GDD $(v = 1 + 2 + n, 3; \lambda_1, \lambda_2)$ exists. They later in [6] found all ordered triples $(n, \lambda_1, \lambda_2)$ of positive integers, with $\lambda_1 > \lambda_2$, such that a GDD $(v = 1 + 2 + n, 3; \lambda_1, \lambda_2)$ exists and completely solved the problem when $k = 3$ on three groups of sizes $(n, 2, 1)$ for any two indices, λ_1 and λ_2 with $\lambda_1 > \lambda_2$ for any n. They show that the necessary conditions are sufficient for the existence of GDD $(n, 2, 1; \lambda_1, \lambda_2)$ for which the indices are both even or else both odd and that if n is odd, the indices must be both even. The same authors in [7] found all ordered pairs (n, λ) of positive integers such that a GDD $(v = 1 + 1 + n, 3; \lambda, 1)$ exists.

Chaiyasena, et al in [2] published a paper in the direction of solving the problem of determining the existence of a GDD $(v = n_1 + n_2 + n_3, 3, \lambda_1, \lambda_2)$ for small values of n_1, n_2, n_3. In particular, for each $n = \{2, 3, 4, 5, 6\}$ they found all ordered pairs (λ_1, λ_2) of positive integers such that a GDD $(v = 1 + 2 + n, 3, 3, \lambda_1, \lambda_2)$ exists.

Lapchinda, et al in [4] found all ordered triples $(n, \lambda_1, \lambda_2)$ of positive integers, with $\lambda_1 < \lambda_2$, such that a GDD $(v = 1 + n + n, 3; \lambda_1, \lambda_2)$ exists and later in [5] considered the problem of determining all ordered triples $(n, \lambda_1, \lambda_2)$ of positive integers, with $\lambda_1 \leq \lambda_2$, such that a GDD $(v = 1 + n + n, 3; \lambda_1, \lambda_2)$ exists by using graph decomposition. They prove that a GDD $(v = 1 + n + n, 3; \lambda_1, \lambda_2)$ exists if and only if a $(2n + 1, 3, \lambda)$-BIBD exists. Whereas the authors in these studies worked on GDDs on three groups of different sizes, they only gave solutions for GDDs of block size three and did not consider the case of block size four.

GDDs of block size four on three groups of different sizes have been studied in very few papers such as in [8] where necessary conditions are proved to be sufficient for the existence of GDD(1, n, n + 1, 4; λ_1, λ_2) whenever $\lambda_1 \geq \lambda_2$ and GDD(2, n, n + 1, 4; λ_1, λ_2) while in [9], the authors proved that the necessary conditions are sufficient for the existence of GDD(3, n, n + 1, 4; λ_1, λ_2). Therefore, the next step is to study a GDD (n_1, n_2, n_3, 4; λ_1, λ_2) where n_1 = 4, n_2 = n and n_3 = n+1 with n $\geq$ 4.

III. Results on existence of GDD (4, n, n+1, 4; λ_1, λ_2)

Modern design theory includes many existence as well as non-existence results. Thus, one fundamental question one may ask about GDD (4, n, n + 1, 4; λ_1, λ_2) under study is: "Does such a GDD exist?" In this section we establish results for the existence of the GDD.

Parameters of GDD (4, n, n + 1, 4; λ_1, λ_2): For GDDs with block size four and three groups of different sizes, 4, n and n + 1, the replication numbers, r_i = 1, 2, 3 are; from $r_1=\frac{3\lambda_1+(2n+1)\lambda_2}{3}$, $r_2 = \frac{(n-1)\lambda_1+(n+5)\lambda_2}{3}$ and $r_3 = \frac{n\lambda_1+(n+4)\lambda_2}{3}$. The GDD has $(n^2+6)\lambda_1$ first associate pairs and $(n^2+9n+4)\lambda_2$ second associate pairs.

The Necessary Conditions for GDD (4, n, n+1, 4; λ_1, λ_2): One of the main items while doing combinatorial design constructions is to find necessary conditions for a particular design. In this subsection, we establish the necessary conditions for a GDD (4, n, n +1, 4; λ_1, λ_2) and prove that they are sufficient.

Theorem 3.1. If a GDD (4, n, n + 1, 4; λ_1, λ_2) exists, then;

i) $3\lambda_1 + (2n + 1)\lambda_2 \equiv 0 \pmod 3$ and $(2n + 1)\lambda_2 \equiv 0 \pmod 3$.

ii) $(n - 1)\lambda_1 + (n + 5)\lambda_2 \equiv 0 \pmod 3$.

iii) $n\lambda_1 + (n + 4)\lambda_2 \equiv 0 \pmod 3$.

iv) $(n^2+6)\lambda_1 + (n^2+ 9n + 4)\lambda_2 \equiv 0 \pmod 6$.

Proof. By counting the replication numbers r_i for elements of the i^{th} group,

i) The replication number for elements in G_1 is obtained from $r_1=\frac{3\lambda_1+(2n+1)\lambda_2}{3}$. Since r_1 is an integer, then $3\lambda_1 + (2n + 1)\lambda_2 \equiv 0 \pmod 3$. Again since $3|3\lambda_1$, it remains that $3| (2n + 1)\lambda_2$, that is $(2n + 1)\lambda_2 \equiv 0 \pmod 3$.

ii) The replication number for elements in G_2 is obtained from $r_2 = \frac{(n-1)\lambda_1+(n+5)\lambda_2}{3}$. Since r_2 is an integer, then $(n - 1)\lambda_1 + (n +5)\lambda_2 \equiv 0 \pmod 3$.

iii) The replication number for elements in G_3 is obtained from $r_3 = \frac{n\lambda_1+(n+4)\lambda_2}{3}$. Since r_3 is an integer, then $n\lambda_1 + (n + 4)\lambda_2 \equiv 0 \pmod 3$.

iv) The number of blocks of the GDD is obtained from $b = \frac{(n^2+6)\lambda_1+(n^2+9n+4)\lambda_2}{6}$. Therefore $(n^2 + 6)\lambda_1 + (n^2 + 9n + 4)\lambda_2 \equiv 0 \pmod 6$.

□

These necessary conditions on b and r_i determine possibilities for the parameter n, and the indices λ_1 and λ_2 which are summarized in Table 1 below where "None" means that the design does not exist for any value of n.

Table 1: The restrictions on n for GDD (4, n, n + 1, 4; λ_1, λ_2)

λ_1 / λ_2	0	1	2	3	4	5	6
0	None	None	None	Any n	None	None	Any n
1	None	n $\equiv$ 4 (mod 6)	None	None	None	None	none
2	None	None	n $\equiv$ 1 (mod 3)	None	None	None	None
3	Even n	None	None	Even N	None	None	Even n

4	None	n $\equiv$ 1 (mod 3)	None	None	n $\equiv$ 1 (mod 3)	None	None
5	None	None	None	None	None	n $\equiv$ 4 (mod 6)	None
6	Any n	None	None	Any n	None	None	Any n

Observations from this table show that there are other necessary conditions as shown below.

1. GDD (4, n, n + 1, 4; λ_1, λ_2) when $\lambda_1, \lambda_2 \equiv 0$ (mod 6).

Example 3.1. A GDD (4, 5, 6, 4; 12, 6) exists with r_1 = 34, r_2 = 36, r_3 = 38 and b = 136. The groups of the GDD are G_1 = {1, 2, 3, 4}, G_2 = {5, 6, 7, 8, 9} and G_3 = {a, b, c, d, e, f}. The design can be constructed taking the 36 blocks of a BIBD (9, 4, 6) on $G_1 \cup G_2$ add 45 blocks of a BIBD (10, 4, 6) on $G_1 \cup G_3$ together with the 55 blocks of a BIBD (11, 4, 6) on $G_2 \cup G_3$.

Theorem 3.2. A GDD (4, n, n + 1, 4; 2λ, λ) exists if a BIBD (4 + n, 4, λ), a BIBD (5 + n, 4, λ) and a BIBD (2n + 1, 4, λ) exist. It is well known that a BIBD (n, 4, 6) exists for n $\geq$ 4 and thus a GDD (4, n, n+1, 4; 12, 6) will always exist. Hence a GDD (4, n, n+1, 4; 12s, 6s) always exists for all positive integers, s.

In general, a GDD (4, 5t, 5t + 1, 4; 12t, 6t) exists with $r_1 = 20t^2 + 14t$, $r_2 = 30t^2 + 6t$, $r_3 = 30t^2 + 8t$ and $b = 25t^3 + 95t^2 + 16t$ where t is a positive integer.

2. GDD (4, n, n + 1, 4; λ_1, λ_2) when $\lambda_1 = \lambda_2$.

A GDD (4, n, n + 1, 4; 0, λ_2) does not exist. This is because there are only three groups and the block size is four. So, each block must contain at least a pair from the same group ($\lambda_1 \geq 1$) to complete the block size. A GDD (4, n, n + 1, 4; λ_1, 0) exists as a (2n + 5, 4, λ_1)-BIBD for particular values of n and λ_1. So, a GDD (4, n, n + 1, 4; 0, 0) does not exist.

Theorem 3.3. A GDD (4, n, n + 1, 4; λ, λ) exists if and only if a (2n + 5, 4, λ)-BIBD exists.

Example 3.2. A GDD (4,10,11,4;1,1) exists as a (25,4,1)-BIBD.
The GDD exists with G_1 = {1, 2, 3, 4}, G_2 = {5, 6, 7, 8, 9, 10, 11, 12, 13, 14} and G_3 = {a, b, c, d, e, f, g, h, i, j, k}. The respective replication numbers are $r_1 = r_2 = r_3 = 8$ and the number of blocks is b = 50. The blocks of the GDD are those of the BIBD. They are:

1	1	1	1	1	1	1	1	2	2	2	2	2	2	2	3	3	3
2	4	5	6	7	8	9	10	4	5	6	7	8	9	11	4	5	6
3	b	12	13	14	c	e	11	14	C	10	f	13	b	12	11	a	d
k	j	g	h	a	d	f	i	i	h	e	g	a	d	j	f	j	i

3	3	3	3	4	4	4	4	4	5	5	5	5	6	6	6	6
7	8	9	10	5	7	8	10	13	7	8	11	14	7	9	12	a
b	e	13	12	6	12	9	a	c	9	10	13	d	8	11	14	b
c	g	14	h	k	e	h	d	g	I	f	b	e	j	g	c	f

7	7	8	8	9	9	10	11	12	13	14	a	b	c	d
10	11	11	12	10	12	14	a	13	e	f	g	e	f	g
13	d	14	b	c	a	b	c	d	i	h	h	h	i	j
k	h	k	i	j	k	g	e	f	j	j	i	k	k	k

To generalize, (4, 10t, 10t+1, 4; t, t) exists for all positive integers, t except for $t \equiv 2 \pmod{3}$.

3. GDD $(4, n, n+1, 4; \lambda_1, \lambda_2)$ when $\lambda_1 < \lambda_2$.

Theorem 3.4. A GDD $(4, n, n+1, 4; \lambda_1 < \lambda_2, \lambda_2)$ exists only when;

i) $\lambda_1 \equiv 0 \pmod{3}$ and $\lambda_2 \equiv 0 \pmod{6}$.

ii) $\lambda_1 \equiv 2 \pmod{3}$ and $\lambda_2 \equiv 2 \pmod{6}$.

Example 3.3. A GDD (4, 4, 5, 4; 3, 6) exists with $r_1 = r_2 = 21$, $r_3 = 20$ and 67 blocks.

In general, GDD (4, 4t, 4t+ 1, 4; 3t, 6t) exists for all positive integers, t.

Example 3.4. A GDD (4, 4, 5, 4; 2t, 8t) exists with $r_1 = r_2 = 26$, $r_3 = 24$ and $b = 93$ blocks.

In fact, a GDD (4, 4t, 4t+1, 4; t, t) exists for all positive integral values of t except $t \equiv 2 \pmod{3}$.

4. GDD $(4, n, n+1, 4; \lambda_1, \lambda_2)$ when $\lambda_1 > \lambda_2$.

Theorem 3.5. A GDD $(4, n, n+1, 4; \lambda_1 > \lambda_2, \lambda_2)$ exists when $\lambda_1, \lambda_2 \equiv 0, 1, 2 \pmod{3}$.

Example 3.5. A GDD (4, 5, 6, 4; 6, 3) exists with $r_1 = 17$, $r_2 = 18$, $r_3 = 19$ and 68 blocks.

Example 3.6. A GDD (4, 4, 5, 4; 4, 1) exists. The groups of the GDD are: $G_1 = \{1, 2, 3, 4\}$, $G_2 = \{5, 6, 7, 8\}$ and $G_3 = \{a, b, c, d, e\}$ with $r_1 = r_2 = 7$, $r_3 = 8$ and 24 blocks.

A GDD (4, 4t, 4t+ 1, 4; 4t, t) exists for all positive integers, t except when $t \equiv 2 \pmod{3}$.

Example 3.7. A GDD (4, 7, 8, 4; 8, 2) exists with $r_1 = 18$, $r_2 = 24$, $r_3 = 26$ and 112 blocks.

In fact, a GDD (4, 7t, 7t+1, 4; 8t, 2t) exists for all positive integral values of t except $t \equiv 2 \pmod{3}$.

Theorem 3.6. Let a GDD $(4, n, n+1, 4; \lambda_1, \lambda_2)$ be a design.

i) If n is even, there is no restriction on the parity of λ_1 and λ_2.

ii) If n is odd, λ_1 must be even.

Proof. The number of blocks, b is given by $b = \frac{(n^2+6)\lambda_1 + (n^2+9n+4)\lambda_2}{6}$.

i) Let n = 4t where $t \geq 1$ is an integer. Then $b = \frac{(16t^2+6)\lambda_1 + (16t^2+36t+4)\lambda_2}{6}$ which implies that λ_1 and λ_2 can be of any parity.

i) Let n = 4t+1 where $t \geq 1$ is an integer. Then $b = \frac{(16t^2+8t+7)\lambda_1 + (16t^2+44t+14)\lambda_2}{6}$ which implies that λ_1 must be even.

Example 3.8. A GDD (4,4,5,4:1,1) exists with n-even, and both λ_1 and λ_2 odd. The groups of the GDD are: $G_1 = \{1, 2, 3, 4\}$, $G_2 = \{5, 6, 7, 8\}$ and $G_3 = \{a, b, c, d, e\}$ with $r_1 = r_2 = r_3 = 4$ and $b = 13$. The blocks are:

1	1	1	1	2	2	2	3	3	4	4	5	6
2	3	5	7	3	6	8	4	7	5	8	a	b
4	a	6	c	5	7	d	6	8	7	A	b	c
b	e	8	d	C	a	e	d	B	e	C	d	e

In general, a GDD (4,4t,4t+1,4;t,t) exists for all positive integers, t except $t \equiv 2 \pmod{3}$.

Example 3.9. A GDD(4, 6, 7, 4; 6, 3) with $r_1 = 19$, $r_2 = 21$, $r_3 = 22$ and $b = 74$, a GDD (4, 4, 5, 4; 12, 6) with $r_1 = 30$, $r_2 = 26$, $r_3 = 32$ and $b = 116$, and a GDD (4, 4, 5, 4; 3, 6) of Example 3.3 all exist with n even and no restriction on the parity of λ_1 and λ_2.

In fact for all positive integral values of t, a GDD (4, 6t, 6t+1, 4; 6t, 3t) with $r_1 = 12t^2 + 7t$, $r_2 = 18t^2$

+ 3t, $r_3 = 18t^2 + 4t$ and $b = 39t^3 + 27t^2 + 8t$, and a GDD (4, 4t, 4t + 1, 4; 12t, 6t) with $r_1 = 16t^2 + 14t$, $r_2 = 20t^2 + 6t$, $r_3 = 24t^2 + 8t$ and $b = 64t^3 + 36t^2 + 16t$ all exist.

Example 3.10. A GDD (4, 5, 6, 4; 6, 3) of Example 3.5 exists with n odd and λ_1 only even.

Theorem 3.7. If a GDD (4, n, n + 1, 4: λ_1, λ_2) exists, then $b \geq \max(2r_i - \lambda_1, 2r_i - \lambda_2)$.

Proof. Consider any first associate pair, say a and b from any group G_i where i = 1, 2, 3. This pair must appear in the design coming together λ_1 times. There are r_i blocks containing each of a and b and $r_i - \lambda_1$ blocks containing only one of them. Thus, the number of blocks must be at least $2r_i - \lambda_1$ to accommodate the pair r_i times. Similarly, consider any second associate pair, say a and c from two different groups G_i. As both of them come together λ_2 times, the number of blocks must be at least $2r_i - \lambda_2$ to accommodate the pair r_i times in the design.

Example 3.11. Consider a GDD (4,5,6,4;6,3) of Example 3.5. In this design, $r_1 = 17$, $r_2 = 18$, $r_3 = 19$ and b = 68.
□

Max $[2r_1 - \lambda_1, 2r_1 - \lambda_2] = \max [2(17) - 6 = 28, 2(17) - 3 = 31] = 31 < 68$

Max $[2r_2 - \lambda_1, 2r_2 - \lambda_2] = \max [2(18) - 6 = 30, 2(18) - 3 = 33] = 33 < 68$

Max $[2r_3 - \lambda_1, 2r_3 - \lambda_2] = \max [2(19) - 6 = 32, 2(19) - 3 = 35] = 35 < 68$

Theorem 3.8. If a GDD (4, n, n + 1, 4: λ_1, λ_2) exists, then $\lambda_1 \geq \frac{(n^2+9n+4)\lambda_2}{5n^2+30}$.

Proof. The design has three groups of size $n_i \geq 4$, the block size. Therefore, each block must have at least one first associate pair. This means that the total number of first associate pairs is at least equal to the number of blocks. Since there are $(n^2 + 6)\lambda_1$ first associate pairs and $\frac{(n^2+6)\lambda_1+(n^2+9n+4)\lambda_2}{6}$ blocks,

$$(n^2+6)\lambda_1 \geq \frac{(n^2+6)\lambda_1+(n^2+9n+4)\lambda_2}{6}$$

$$5(n2+6)\lambda_1 \geq (n2+9n+4)\ )\ \lambda_2$$

$$\lambda_1 \geq \frac{(n^2+9n+4)\lambda_2}{5n^2+30}.$$

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Corollary 1. If a GDD (4, n, n + 1, 4; λ_1, λ_2) exists, then $b \leq (n^2 + 6)\lambda_1$.

Proof. The design has b blocks and $(n^2 + 6)\lambda_1$ first associate pairs. The total number of blocks cannot exceed the total number of first associate pairs. Thus $b \leq (n^2 + 6)\lambda_1$.
□

Example 3.14. A GDD (4,5,6,4;12,6) of Example 3.1 exists with 372 first associates and 136 blocks.

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